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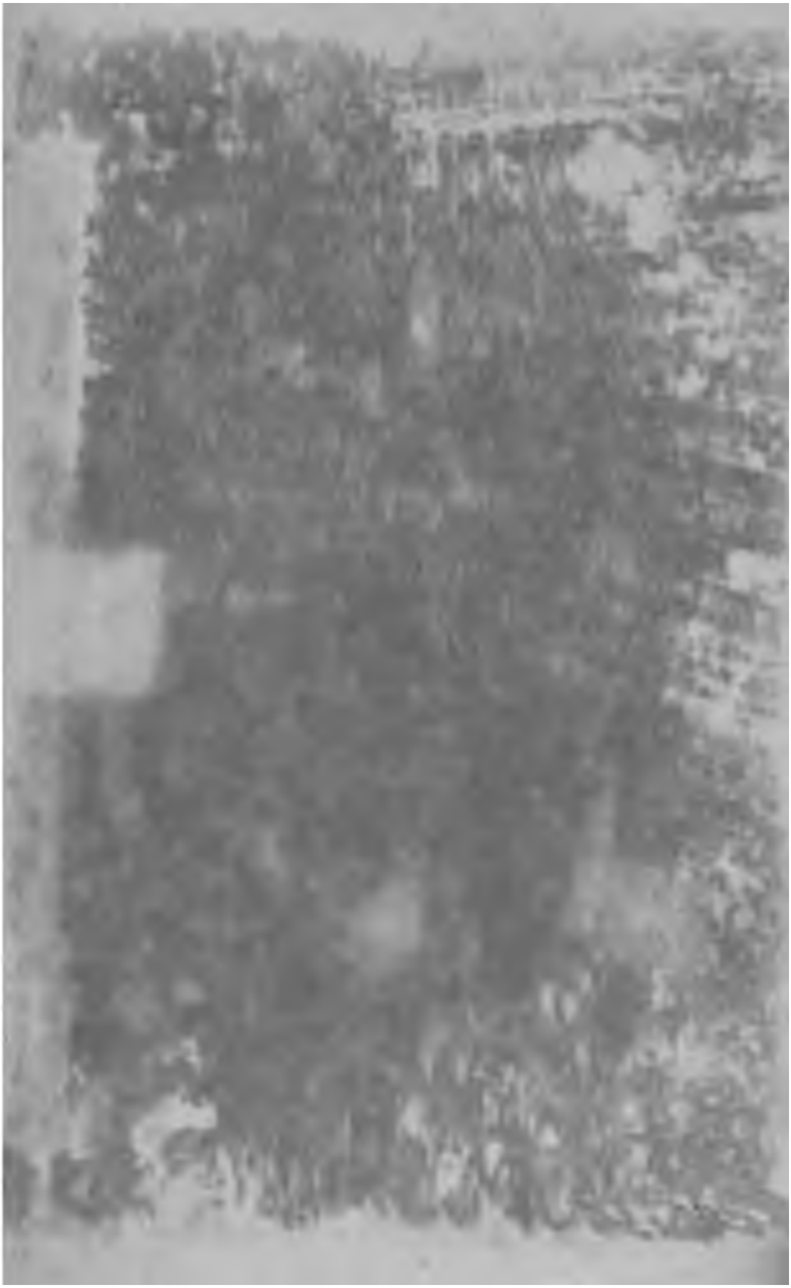

THE DIFFICULTIES OF

ALGEBRA MADE EASY;

A BOOK FOR BEGINNERS.

BY

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PREFACE.

During some experience in the teaching of Elementary Mathematics, it has continually occurred to the author of the following pages that Algebra is more often regarded by a learner as useless drudgery than as a charming and invaluable study. This, however, is not at all times the fault of the beginner himself, but may not unfrequently be ascribed to a serious deficiency in the teaching. When it is placed before one as a mechanical and disjointed subject, it must inevitably appear dull and burdensome; but when taught in a *scientific* and connected manner, its beauties become at once apparent; and, in place of wearisomeness and dislike, interest and admiration will manifest themselves.

The annual examinations of the Science and Art Department have done much towards placing this subject before beginners in its proper light, and the cases have not been few, in which preparation for them has been the means by which even more advanced students have first become acquainted with its great and genuine merits. It is especially for pupils, studying for these examinations, that this little work has been prepared; but at the same time it is hoped that by very many others its advantages will be embraced, and its usefulness appreciated.

The communication of any ideas for its improvement, that may occur to teachers, will be esteemed a favour: and here the author must acknowledge, with many thanks, the exceeding kindness of the friend and able mathematician by whom each page of the MS. has been so carefully and generously examined.

In placing these few pages in the hands of pupils, it must not for one moment be supposed that it has been intended, by their use, to supersede larger and more general works. Not in the least. The object has been to collect in a small and compact form the simple *science* of the early parts of the subject, and to place it before the learner in such a way that this little work may not only be a companion to his other Algebra, but may also easily enable him intelligently to read up that portion which *must* be thoroughly mastered, in order that real proficiency may be attained.

It now only remains to test the usefulness of the work by trial, and this, it is confidently expected, will procure a verdict in its favour.

H. C. TARN.

Devonport,
Sept. 17*th*, 1874.

THE DIFFICULTIES OF ALGEBRA MADE EASY.

CHAPTER I.

Introductory Remarks.

1. In commencing the study of this interesting and important subject, it is highly necessary that the student should thoroughly clear the ground as he advances, and make himself perfect master of the various terms and steps that will occur in the very beginning of its most elementary parts.

There should be no confusion in his mind as to the use of signs; he should clearly comprehend the meaning of such technicalities as factor, term, power, index, coefficient, &c., &c.; the intricacies of brackets should be made simple; and the reason for changing all the signs inside a bracket having a minus sign before it, when the bracket is removed, should be completely mastered. Proofs of rules, and of various formulæ that need proofs, must be firmly fixed on the mind; and such steps as changing all the signs of the subtrahend in subtraction, and then proceeding as in addition, not only remembered and acted upon accordingly, but made so clear that the student himself could, without doubt or hesitancy, explain the process to another.

2. It has not been thought necessary in this early stage of the subject to introduce proofs and definitions, as they may be so completely read up by the student in any ordinary elementary Algebra, but we shall now come to a part which will demand much more attention, and have to bring to the notice of the reader many important formulæ, the value of which it is impossible to over-estimate. Various methods of application will be shown, and it will be afterwards found by any one who completely masters their complicated but yet not difficult uses, that he has within his grasp one of the most powerful levers that can be used, for the complete removal of the numerous difficulties that would otherwise occur.

3. After much experience in teaching, it has been found that the application of these formulæ constitutes one of the very greatest impediments that the beginner has to overcome; whilst on account of their apparent difficulty, students are often pushed on to more advanced parts, without mastering this "*pons asinorum*" of Algebra, and are therefore constantly obliged to perform a deal of mechanical drudgery which an intelligent use of the formulæ would entirely prevent.

4. We will premise their introduction by remarking that all those which will be herein mentioned are the results of multiplication, and by asking the student to pass over no single step until he is as satisfied that each successive one is clearly deducible from its preceding, as he is that $8+4-3=9$.

CHAPTER II.

The square of a binomial.—Examples.—The square of a trinomial, &c.—Examples.—Changing the form of an expression.—The Form of a perfect square.—Example.

5.

$$\begin{array}{ll} a+b & \\ a+b & \\ \hline a^2+ab & \\ \phantom{a^2+{}}ab+b^2 & \\ \hline a^2+2ab+b^2. & \\ \hline\hline \end{array}$$

It is here shown by actual multiplication that $(a+b)\ (a+b)$ or $(a+b)^2=a^2+2ab+b^2$. This latter result then is the formula to be noticed, and by its careful and intelligent application, the square of any binomial may be readily ascertained.

6. If we name the terms according to their position, $(a+b)^2 = a^2+2ab+b^2$ may be written thus, $(1st+2nd)^2=1st^2+2\times 1st\times 2nd+2nd^2$, and we may say that the square of any binomial equals

the square of the first . . . *or* $1st^2$,
the square of the second. . . ,, $2nd^2$,
and twice the first into the second ,, $2\times 1st\times 2nd$.

7. The student should familiarize himself with this identity, and then find the squares of

1. $a+c$.
2. $d+e$.
3. $g+h$.
4. $m+n$.
5. $p+q$.
6. $m+q$.
7. $13+9$.
8. $11+8$.
9. $17+18$.
10. $15x+13y$.
11. $7y^2+8z$.
12. $ax+by$.
13. $pq+2s$.
14. x^2+y^2.
15. x^3+2y^3.
16. $6xy+4ab$.

8. It must be clearly borne in mind that none of these examples must be worked by actual multiplication, but by taking

the square of the first,
the square of the second,
and twice the first into the second.

Thus, $(6xy+4ab)^2$
$=$ the square of the first $(36x^2y^2)$,
$+$ the square of the second $(16a^2b^2)$,
$+$twice the first into the second $(2\times 6xy\times 4ab) = 48abxy$.
$\therefore\ (6xy+4ab)^2 = 36x^2y^2+16a^2b^2+48abxy$.

9.
$$\begin{array}{l} a-b \\ \underline{a-b} \\ a^2-ab \\ \underline{\quad\;\, -ab+b^2} \\ \quad a^2-2ab+b^2. \end{array}$$

After going carefully through the first formula, and working its examples in strict accordance with the mental process contained therein, the above identity needs not to be enlarged upon. The signs, however, deserve much attention, and it will be well to square the few following binomials for the purpose of creating familiarity with them, bearing in mind in so doing, that the signs must be prefixed to their terms, exactly as they result from the multiplication.

17. $ab-cd$. 18. x^3-y^3. 19. $-xy+z$.
20. $-y+xz$. 21. $-2z-4y$. 22. $-4b-6cd$.
23. $-mp+2nq$. 24. $-7mx-3st$.

10. We shall now give an extension of the above formulæ, and show how easily they may be applied to other and more important cases. For example, suppose we require to square the trinomial $x+y-z$,

we may put p for $y - z$, and proceed thus—
$(x+y-z)^2 = (x+p)^2 = x^2 + p^2 + 2px.$

Here we must substitute the $y - z$ for the p, and say—
$x^2 + p^2 + 2px = x^2 + (y-z)^2 + 2x(y-z)$
$= x^2 + y^2 + z^2 - 2yz + 2xy - 2xz.$

11. This process of substitution, however, needs not to be resorted to, as the peculiar faculty of brackets to convert any number of terms into one quantity may be more conveniently employed. Thus:
$(x+y-z)^2 = \{(x+y) - z\}^2.$

Here $(x+y)$ is the first term of the binomial $\{(x+y) - z\}^2$, and must be subjected to all the processes which were previously applied to the a in $(a+b)^2$.
$\because \{(x+y) - z\}^2 = (x+y)^2 + z^2 - 2(x+y)z$
$= x^2 + y^2 + 2xy + z^2 - 2xz - 2yz.$
$= x^2 + y^2 + z^2 + 2xy - 2xz - 2yz.$

(N.B. This last step has been taken for the sake of reference.)

Again $(a - b - c + d)^2$
$= \{(a-b) - (c-d)\}^2 = (a-b)^2 + (c-d)^2 - 2(a-b)(c-d)$
$= a^2 + b^2 - 2ab + c^2 + d^2 - 2cd - 2ac + 2ad + 2bc - 2bd$
$=$ (as above) $a^2 + b^2 + c^2 + d^2 - 2ab - 2ac + 2ad$
$+ 2bc - 2bd - 2cd.$

12. It will be noticed from the last two expansions, that there is in them a similar and peculiar arrangement of terms, and this arrangement may be fully relied on as a formula for squaring *any* quantity, whether it be a binomial, a trinomial, or a multinomial. By carefully examining them the student will discover that the law which governs the expansion may be stated thus:—

The square of each term,
and the double of each term into every one that follows it;

so that $(x+y-z+p-q)^2$
$=x^2+y^2+z^2+p^2+q^2+2xy-2xz+2px-2qx-2yz$
$+2py-2qy-2pz+2qz-2pq.$

13. The learner should carefully work out many examples for himself, and judiciously employ the various ways that have been proposed. He must remember that the position of the terms in the answer has no effect whatever on the value.

25. $(a+b-c^2)^2$. 26. $(a+x-2b-2y)^2$.
27. $(x-y-6+z)^2$. 28. $(2a+3b+4c+5d)^2$.
29. $(p-2q+r-3m)^2$.
30. $(1-2a+3a^2-2a^3+a^4)^2$.
31. $(x^4+x^3y-x^2y^2-xy^3+2y^4)^2$.
32. $\{(a+x)^2\}^2$. 33. $\{(2b+3c)^2\}^2$.
34. $[\{(x+y)^2\}^2]^2$.
35. $[a+b-\{c-d-(a+b)\}]^2$.
36. $[-(a+b)-\{c-(a-d)+x\}+y]^2$.
37. $(\frac{x}{a}+\frac{a}{x})^2$. 38. $(1+\frac{x}{y}+\frac{y}{x})^2$.
39. $(\frac{x+a}{x-a}+\frac{x-a}{x+a})^2$.

14. Again suppose it is required to write down the quotient of

$$(x+y)^2+2(x+y)z+z^2 \div \overset{x}{z}+y+z.$$

It is not here at all necessary to make a long division sum of the work, but careful observation will discover that in the dividend there is

the square of a first,
the square of a second,
and twice the first into the second;

the first being $(x+y)$, the second z, and the product of course $2(x+y)z$; so that the expression

$$(x+y)^2+2(x+y)z+z^2$$

is at once written down as $\{(x+y)+z\}^2$.

∴ The problem runs thus

$$\frac{(x+y)^2+2(x+y)z+z^2}{x+y+z}=\frac{\{(x+y)+z\}^2}{x+y+z}$$
$=x+y+z.$

15. The superiority of ready *mental* work over *mechanical* is so great, and the advantages involved so numerous, that even in the comparatively few cases where the former may be the longer, it should certainly have the preference.

16. It is a very useful exercise in Algebra, for the student to change an expression from one form to another. Thus let him shew that

$(a+b)^2+2(a+b)(a-b)+(a-b)^2=(2a)^2.$

This identity, by a beginner, would generally be proved as follows:—

$(a+b)^2+2(a+b)(a-b)+(a-b)^2 = a^2+2ab+b^2+2a^2-2ab+2ab-2b^2+a^2-2ab+b^2=4a^2$ $=(2a)^2$; and, of course, the required result would be obtained; but the method employed is far inferior to

$$(a+b)^2+2(a+b)(a-b)+(a-b)^2$$
$$=\{(a+b)+(a-b)\}^2=(2a)^2.$$

Indeed the very form in which the right hand member of the identity appears in the original, viz., as $(2a)^2$, would suggest that the latter method of solution was the one intended.

17. The learner must carry continually in his mind the *form of a perfect square*, as he will then be enabled, without difficulty, to collect into some definite form, certain quantities which may otherwise less readily convey an impression of their nature. Thus, show that

$(a+b+c)^2+a^2+b^2+c^2 = (a+b)^2+(b+c)^2+(c+a)^2.$

In working such examples as these, the left-hand member must be taken as it stands, and gradually changed until it easily assumes the form of the right-hand member—this latter being altogether unused throughout the operation. Or, the process may be reversed.

$$
\begin{aligned}
\therefore\ & (a+b+c)^2+a^2+b^2+c^2 \\
=\ & \{(a+b)+c\}^2+a^2+b^2+c^2 \\
=\ & (a+b)^2+2(a+b)c+c^2+a^2+b^2+c^2 \\
=\ & (a+b)^2+2ac+2bc+c^2+a^2+b^2+c^2 \\
=\ & (a+b)^2+(b^2+2bc+c^2)+(c^2+2ac+a^2) \\
=\ & (a+b)^2+(b+c)^2+(c+a)^2.
\end{aligned}
$$

N.B.—The $(a+b)^2$ *is retained unaltered since it occurs in the same form in the answer.*

18. Other examples may readily be found in any good Elementary Algebra for the purpose of facilitating the use and application of these formulæ.

CHAPTER III.

The product of a sum and difference.—Examples.—Inversion of $(a+b)(a-b) = a^2 - b^2$.—*Examples.—Extended application of* $(a+b)(a-b) = a^2 - b^2$.—*Fully-worked illustrative examples.—Resolution into Factors.—Examples.*

19.

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad\ -ab-b^2 \\ \hline a^2\quad\ -b^2 \\ \hline \end{array}$$

The next theorem for consideration is

$$(a+b)(a-b) = a^2 - b^2.$$

This is a very important proposition, and enters largely into the formation of Algebraic expressions. The amount of work that may be saved by its application is almost beyond belief, whilst the most unlikely-looking quantities are often with care converted into the form of one or the other of its parts.

20. The formula may be thus expressed in words: *When the sum of two quantities is multiplied by the difference of the same two, the product equals the difference of their squares.*

This should be carefully learnt by heart and thoroughly understood. Multiplications which would otherwise be long and wearisome, may by the aid of this invaluable formula, be easily and quickly performed. The student has simply to arrange his factors so that the first and second terms of each are respectively the same, but the signs different. If the expressions are capable of arrangement in this manner, the product is found by squaring each term, and putting a minus sign between the two results. At the same time it must be carefully borne in mind that *any number of terms may be inside a bracket and thus practically form only* ONE *term.* For instance, $a+b-c+d$ consists of four terms, $(a+b)-(c-d)$ of two terms, while $(a+b-c+d)$ may be regarded as one term.

21. We shall now proceed to apply the formula under notice. It will readily be seen that $(x+y)(x-y) = x^2 - y^2$, and $(m+n)(m-n) = m^2 - n^2$, because they so clearly correspond to $(a+b)\ (a-b) = a^2 - b^2$. Therefore after the learner has worked a few such examples for himself, he will be able to proceed at once to more important and complicated ones.

1. $(s+2t)(s-2t)$.
2. $(x-c)(x+c)(x^2+c^2)$.
3. $(x-2r)(x+2r)(x^2+4r^2)$.
4. $(2p-q)(q+2p)$.
5. $(a^2x-x^3)(a^2+x^2)$.
6. $(\frac{7x}{2b}+\frac{3y}{4c})(\frac{7x}{2b}-\frac{3y}{4c})$.
7. $\{(a^2-x)(a^2+x)\}^2$.
8. $\{(x-a)(x+a)(x^2+a^2)(x^4+a^4)\}^2$.

22. Inverting the above process, of course it will be instantly perceived that if $(a+b)\ (a-b) = a^2 - b^2$,

$$\therefore a^2 - b^2 = (a+b)(a-b).$$

Thus many quantities may easily be broken into elementary factors, and much unnecessary work be often saved. For example, *every binomial that has each term a perfect square, with a minus sign between the two terms* is capable of the above resolution.

$\therefore 9x^2 - 4y^2 = (3x+2y)(3x-2y)$.

The student should carefully remember and imitate this breaking into factors, by resolving any appropriate examples. Subjoined are a few.

9. $x^4 - 81y^2$. 10. $x^8 - 1$. 11. $x^4 - 64$.

12. $1 - 225x^4$. 13. $\frac{x^4}{a^2} - \frac{y^4}{b^2}$. 14. $(a^2 - b^2)^2$.

15. $a^2x^2 - a^2y^2$. 16. $\frac{1}{4}a^2 - \frac{1}{9}b^2$. 17. $x^4y^4z^4 - 1$.

18. $256x^4 - 625y^4$. 19. $a^4b^{12} - c^8$.

20. $am^2p - an^4p$.

23. By aid of this formula many examples in division may be readily performed, and the G.C.M. of two or more quantities be often found easily by inspection. Thus,

$$x^8 - 81y^8 \div x^2+3y^2 = \frac{(x^4 - 9y^4)(x^4+9y^4)}{x^2 + 3y^2}$$

$$= \frac{(x^2 - 3y^2)(x^2+3y^2)(x^4+9y^4)}{x^2+3y^2}$$

$$= (x^2-3y^2)(x^4 +9y^4).$$

Again the G.C.M. of

$a^2b - ab^2$, $a^2 - b^2$, and $(a-b)^2$, is at once discovered to be $a - b$.

24. Numerous other instances of the value of the above formulæ will continually occur in practice, indeed it would be quite impossible within the limits of a small book, to exhaust the examples that any good Algebra will furnish.

25. We will now proceed to apply

$$(a+b)(a-b) = a^2 - b^2$$

in an extended manner, and to introduce some important and note-worthy illustrations.

26. It is required to find the product of

$$(a \quad b - c)(a+b+c).$$

Here the factors may be arranged as follows:

$$(a+b) - c\}\{(a+b)+c\},$$

and thus be made to correspond to $(x+y)(x-y)$, since the first and second terms in both are respectively alike, and the signs different.

$$\therefore\ (a+b+c)(a+b-c)=\{(a+b)+c\}\{(a+b)-c\}$$
$$=(a+b)^2-c^2=a^2+2ab+b^2-c^2.$$

27. The student must carefully follow for himself the arrangement and solution of the following examples, and this he will have no difficulty whatever in doing, provided he has thoroughly mastered the preceding work.

(1.) $(a+b+c-d)(a+b-c+d)$

$$\begin{aligned}&=\{(a+b)+(c-d)\}\{(a+b)-(c-d)\}\\&=(a+b)^2-(c-d)^2\\&=a^2+2ab+b^2-c^2+2cd-d^2.\end{aligned}$$

(2.) $(x+2m-3z)\ (x-2m+3z)$

$$\begin{aligned}&=\{x+(2m-3z)\}\{x-(2m-3z)\}\\&=x^2-(2m-3z)^2\\&=x^2-4m^2+12mz-9z^2.\end{aligned}$$

(3.) $(x^2-ax+a^2)(x^2+ax+a^2)(x^4-a^2x^2+a^4)$

$$\begin{aligned}&=\{(x^2+a^2)-ax\}\{(x^2+a^2)+ax\}(x^4-a^2x^2+a^4)\\&=(x^4+a^2x^2+a^4)(x^4-a^2x^2+a^4)\\&=\{(x^4+a^4)+a^2x^2\}\{(x^4+a^4)-a^2x^2\}\\&=(x^4+a^4)^2-a^4x^4\\&=x^8+2a^4x^4+a^8-a^4x^4\\&=x^8+a^4x^4+a^8.\end{aligned}$$

(4.) $(p^4-2p^3+3p^2-2p+1)\ (p^4+2p^3+3p^2+2p+1)$

$$\begin{aligned}&=\{(p^4+3p^2+1)-(2p^3+2p)\}\{(p^4+3p^2+1)\\&\quad+(2p^3+2p)\}\\&=(p^4+3p^2+1)^2-(2p^3+2p)^2\\&=p^8+9p^4+1+6p^6+2p^4+6p^2-4p^6-8p^4-4p^2\\&=p^8+2p^6+3p^4+2p^2+1.\end{aligned}$$

28. What has previously been said relative to the breaking up of expressions resembling (a^2-b^2)

into factors resembling $(a+b)(a-b)$, is, of course, applicable to the present extensions; and the results, though more complicated, are far more interesting. We will give a few examples, each of which should be thoroughly analysed and understood.

(1.) $a^2+2ab+b^2-c^2$
$= (a+b)^2-c^2$
$= (a+b+c)(a+b-c)$.

N.B.—Since the expression $(a+b)^2-c^2$ is the difference of two perfect squares, therefore it resembles x^2-y^2, and consequently can be made to resemble $(x+y)\ (x-y)$, the two factors into which x^2-y^2 is divisible.

(2.) $(a+b-c)(a-b+c)(b+c-a) \div (a^2-b^2-c^2+2bc)$

$$= \frac{(a+b-c)(a-b+c)(b+c-a)}{\{a^2-(b^2-2bc+c^2)\}}$$

$$= \frac{(a+b-c)(a-b+c)(b+c-a)}{(a-b+c)(a+b-c)}$$

$= b+c-a$.

(3.) $ax^2-ab^2+b^2x-x^3 \div (x+b)(a-x)$

$$= \frac{a(x^2-b^2)-x(x^2-b^2)}{(x+b)(a-x)}$$

$$= \frac{(a-x)(x^2-b^2)}{(x+b)(a-x)}$$

$$= \frac{(a-x)(x-b)(x+b)}{(x+b)(a-x)} = x-b\ ^*.$$

(4.) Resolve $4x^2y^2-(x^2+y^2-z^2)^2$ into four factors.

$4x^2y^2-(x^2+y^2-z^2)^2$
$= \{2xy-(x^2+y^2-z^2)\}\{2xy+(x^2+y^2-z^2)\}$

---

[*] The *collection of factors* employed in this and other examples will be afterwards explained. (See 53.)

$$= (2xy - x^2 - y^2+z^2)(2xy+x^2+y^2 - z^2)$$
$$= \{z^2-(x^2 -2xy+y^2)\}\{(x^2+2xy+y^2)-z^2\}$$
$$= (z - x+y)(z+x - y)(x+y - z)(x+y+z).$$

(5.) Show that $(p^2+pq+q^2)^2 - (p^2 - pq+q^2)^2 = 4pq(p^2+q^2)$.

$$(p^2+pq+q^2)^2 - (p^2 - pq+q^2)^2$$
$$= \{(p^2+pq+q^2) - (p^2 - pq+q^2)\}\{(p^2+pq+q^2) + (p^2 - pq+q^2)\}$$
$$= (p^2+pq+q^2-p^2+pq-q^2)(v^2+pq+q^2+p^2-pq+q^2)$$
$$=2pq(2p^2+2q^2)$$
$$= 4pq(p^2+q^2).$$

(6.) Find the value of $x^4 - y^4+(x^2 - y^2)^2 -2x^4+2x^2y^2$.

$$x^4 - y^4+(x^2 - y^2)^2 - 2x^4+2x^2y^2$$
$$= (x^2 - y^2)(x^2+y^2)+(x^2 - y^2)(x^2 - y^2) - 2x^2(x^2 - y^2)$$
$$= (x^2 - y^2)(x^2+y^2+x^2 - y^2 - 2x^2)$$
$$= (x - y)(x+y) \times 0 = 0.$$

(7.) $a^2 - \left(\dfrac{a^2+b^2 - c^2}{2b}\right)^2$

$$= a^2 - \frac{(a^2+b^2 - c^2)^2}{4b^2}$$
$$= \frac{4a^2b^2 - (a^2+b^2 - c^2)^2}{4b^2}$$
$$= \frac{(2ab-a^2-b^2+c^2)(2ab+a^2+b^2-c^2)}{4b^2}$$
$$= \frac{\{c^2 - (a^2 -2ab+b^2)\}\{(a^2+2ab+b^2) -c^2\}}{4b^2}$$
$$= \frac{(c - a+b)(c+a - b)(a+b - c)(a+b+c)}{4b^2}.$$

29. Similar examples may be supplied, *ad infinitum*, but we will now subjoin a few important fractions involving the application of these useful formulæ.

(1.) Reduce $\frac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a^4}$ to lowest terms.

$$\frac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a^4}$$

$$=\frac{(x^4+2a^2x^2+a^4)-a^2x^2}{(x^4-a^4)+ax(x^2-a^2)}$$

$$=\frac{(x^2+a^2)^2-a^2x^2}{(x^2-a^2)(x^2+a^2)+ax(x^2-a^2)}$$

$$=\frac{(x^2+a^2-ax)(x^2+a^2+ax)}{(x^2-a^2)(x^2+a^2+ax)}$$

$$=\frac{x^2-ax+a^2}{x^2-a^2}.$$

(2.) $\frac{(x+y)^2-(p+q)^2}{(x+p)^2-(y+q)^2}$

$$=\frac{(x+y-p-q)(x+y+p+q)}{(x+p-y-q)(x+p+y+q)}$$

$$=\frac{x+y-p-q}{x+p-y-q}.$$

(3.) $\frac{a^2+x^2+2ax-z^2}{z^2-a^2-x^2+2ax} \div \frac{a+x+z}{x+z-a}$

$$=\frac{(a+x)^2-z^2}{z^2-(a-x)^2} \div \frac{a+x+z}{x+z-a}$$

$$=\frac{(a+x-z)(a+x+z)}{(z-a+x)(z+a-x)} \times \frac{x+z-a}{a+x+z}$$

$$=\frac{a+x-z}{z+a-x}.$$

30. The above applications will certainly be sufficient to give the student some little idea of the manner in which he should examine any examples that may occur, for the purpose of introducing into his work, if possible, the important aid of these formulæ. A few exercises are subjoined which very plainly in-

volve their use.

21. Find the L.C.M. of y^2-4, y^4-16, y^2-9, $y+3$.

22. $(a-b+c)(c+b-a)(a^2+b^2+c^2-2ab)$.

23. $(\frac{a^4}{b^4}-\frac{b^4}{a^4}) \div (\frac{a}{b}-\frac{b}{a})$.

24. $(a-b)^2(a+b)^2(a^2+b^2)^2$.

25. $\frac{1}{a^2-ab+b^2} \times \frac{1}{a^2+ab+b^2} \times \frac{1}{a^4-a^2b^2+b^4}$.

26. $x^8-y^8+x^6y^2-x^2y^6 \div (x^2-xy+y^2)(x^2+xy+y^2)$.

27. Resolve $x^{16}-65536$ into five factors.

23. Multiply $1+\frac{x}{1-x}$ by $1-\frac{x}{1-x}$.

29. $\frac{x^4-y^4}{x^2-2xy+y^2} \times \frac{x-y}{x^2+xy}$.

30. $(a+x)^4-\frac{1}{(a+x)^4} \div (a+x)^2+\frac{1}{a^2+x^2+2ax}$.

31. $(\frac{a^2}{p^2}+\frac{x^2}{y^2}-1)\ (\frac{a^2}{p^2}-\frac{x^2}{y^2}+1)$.

32. $(x+\sqrt{x^2+9})(x-\sqrt{x^2+9})$.

33. Multiply $a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b$ by $a-a^{\frac{1}{2}}b^{\frac{1}{2}}+b$.

34. Resolve $(a+b)^{4n}-c^{4n}$ into elementary factors.

CHAPTER IV.

The sum of two perfect Cubes.—The difference of two perfect Cubes.—Extended Application of Formulæ. Fully-worked illustrative examples.—Important Factorial Resolutions.—Resolution by Fractional Indices.—Easy Deductions from Formulæ.—Useful Abstract for Memory.

31.

$$\begin{array}{l}
a^2 - ab+b \\
\underline{a + b} \\
a^3 - a^2b + ab^2 \\
\quad\underline{+ a^2b - ab^2+b^3} \\
\underline{\underline{a^3 \qquad\qquad\qquad +b^3}}
\end{array}$$

$$\begin{array}{l}
a^2+ab+b^2 \\
\underline{a - b} \\
a^3+a^2b+ab^2 \\
\quad\underline{-a^2b - ab^2 - b^3} \\
\underline{\underline{a^3 \qquad\qquad\qquad - b^3}}
\end{array}$$

$\therefore\ a^3+b^3 = (a+b)(a^2 - ab+b^2)$,
and $a^3 - b^3 = (a - b)(a^2+ab+b^2)$.

Here again are two very important formulæ, and the student must carefully notice and learn the order and nature of the factors. From what has been said concerning the preceding identities, there is no necessity to enter into a written description of each; suffice it to say, that the binomial factors resemble, *in signs*, the terms of the original products. Numerous examples involve these formulæ in combination with others, as many of the following illustrations will show.

32. With regard to any of the identities which will be included in these remarks, it is scarcely too much to say that there is hardly any Algebraic pro-

blem to be solved, or theorem to be proved, which should not be carefully analysed for the discovery of some hidden formula; and this, whenever it does occur, will invariably shorten and simplify the work. The careful attention of the student is called to the following examples.

(1.) Resolve $(x+y)^3+c^3$ into factors.

This example evidently resembles a^3+b^3, since both the terms are perfect cubes, and they are united by a plus sign.

$\therefore$ The factors will resemble

$$(a+b)\ (a^2-ab+b^2).$$

Now then, let this order of terms, indices, and signs be imitated exactly, and the factors of the given expression will be obtained. It must be borne in mind that $(x+y)$, in $(x+y)^3+c^3$, stands for a in a^3+b^3, and will therefore be the root of the perfect cube which forms the first term of the binomial expression to be resolved. It remains now only to see clearly that $(x+y)^3+c^3$

$$=(x+y+c)\{(x+y)^2-(x+y)c+c^2\},$$

and one good step will be made towards the complete understanding of the many applications that may be adduced.

(2.) Divide $a^6+b^6-2a^3b^3$ by $(a-b)^2$.

This may, of course, be worked by ordinary division, but such a method becomes absolutely wearisome after the prettier and more scientific use of formulæ. Thus,

$$\frac{a^6+b^6-2a^3b^3}{(a-b)^2}$$

$$=\frac{(a^3-b^3)^2}{(a-b)^2}$$

$$= \frac{(a-b)^2(a^2+ab+b^2)^2}{(a-b)^2}$$

$$= (a^2+ab+b^2)^2.$$

All the recommendation needed for the above is to understand it.

(3.) $$\frac{1+8x^3}{1+2x}=\frac{1^3+(2x)^3}{1+2x}$$

$$\frac{(1+2x)(1-2x+4x^2)}{1+2x}$$

$$= 1-2x+4x^2.$$

Here $8x^3$ being the second term, and a perfect cube, corresponds to the b^3 in a^3+b^3; and, consequently, $2x$, the $\sqrt[3]{8x^3}$, stands for the b in $a+b$, the first of the two factors into which a^3+b^3 is divisible. The 1 stands both for a^3 and a, since 1 is the $\sqrt[3]{1}$.

(4.) $$\frac{x^8-y^8+x^2y^2(x^4-y^4)}{(x^2-xy+y^2)(x^2+xy+y^2)}$$

$$= \frac{(x^4-y^4)(x^4+y^4)+x^2y^2(x^4-y^4)}{\{(x^2+y^2)-xy\}\{(x^2+y^2)+xy\}}$$

$$= \frac{(x^4-y^4)(x^4+y^4+x^2y^2)}{x^4+y^4+2x^2y^2-x^2y^2}$$

$$= \frac{(x^4-y^4)(x^4+y^4+x^2y^2)}{x^4+x^2y^2+y^4}$$

$$= x^4-y^4.$$

(5.) $\frac{1}{8}(x+y)^3+c^3 \div \frac{1}{2}(x+y)+c$

$$= \frac{\{\frac{1}{2}(x+y)+c\}\{\frac{1}{4}(x+y)^2-\frac{1}{2}(x+y)c+c^2\}}{\frac{1}{2}(x+y)+c}$$

$$= \tfrac{1}{4}(x+y)^2-\tfrac{1}{2}(x+y)c+c^2.$$

This is a good example, and should be carefully studied.

(6.) $$\frac{m^{18}-n^{12}}{m^3+n^2}=\frac{(m^6-n^4)(m^{12}+m^6n^4+n^8)}{m^3+n^2}$$

$$= \frac{(m^3 - n^2)(m^3+n^2)(m^{12}+m^6n^4+n^8)}{m^3+n^2}$$

$$= (m^3 - n^2)(m^{12}+m^6n^4+n^8).$$

It is not so easy to see at first that $m^{18} - n^{12}$ corresponds to $a^3 - b^3$, but it reveals its similarity as soon as it is observed that each of the terms is a perfect cube, and that the expression may read thus,

$$(m^6)^3 - (n^4)^3,$$

where m^6 stands for a, and n^4 *for* b.

(7.) $$\frac{p^3+q^3+pq(p^2 - q^2)+q^3(p+q)}{(1+q)(p+q)}$$

$$= \frac{(p+q)(p^2-pq+q^2)+pq(p-q)(p+q)+q^3(p+q)}{(1+q)(p+q)}$$

$$= \frac{(p+q)(p^2 - pq+q^2+p^2q - pq^2+q^3)}{(1+q)(p+q)}$$

$$= \frac{p^2 - pq+q^2+p^2q - pq^2+q^3}{1+q}$$

$$= \frac{p^2(1+q) - pq(1+q)+q^2(1+q)}{1+q}$$

$$= \frac{(1+q)(p^2 - pq+q^2)}{1+q} = p^2 - pq+q^2.$$

The method adopted to get rid of $1+q$, is simply suggested by seeing the $1+q$ remaining in the denominator. This, of course, can only be done by getting the $1+q$ to appear as a factor in the numerator, the process for which is very plain and simple.

33. The very fact that with any of the formulæ there is a product of factors, together with the factors themselves, is sufficient evidence that expressions of which the G.C.M. or L.C.M. is required, must involve their constant use. By the solution of Algebraic quantities into their elementary parts, many otherwise rather long examples may be readily and easily solved.

(8.) Find the G.C.M. of (y^2-y+1) and $(y^6-1$.
$y^6-1=(y^3+1)(y^3-1)=(y+1)(y^2-y+1)(y^3-1)$.
$\therefore\ y^2-y+1=$ the G.C.M., by inspection.

(9.) Find the L.C.M. of y^2-1, y^3+1 and y^3-1.
$y^2-1=(y-1)(y+1)$;
$y^3+1=(y+1)(y^2-y+1)$;
$y^3-1=(y-1)(y^2+y+1)$;
$\therefore$ The L.C.M. $=(y+1)(y^2-y+1)(y-1)(y^2+y+1)$
$=(y^3+1)(y^3-1)=y^6-1$.

34. When two terms, separated by a minus sign, have each an index which is the sum of *two* equal quantities, and also of *three* (as, for example, 6, which $=2+2+2$ and $3+3$) the expression is capable of resolution into factors by the application either of a^2-b^2 or a^3-b^3.

Thus,

$$x^6-y^6=(x^3+y^3)(x^3-y^3)$$
$$=(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$$
$$=(x+y)(x-y)\{(x^2+y^2)-xy\}\{(x^2+y^2)+xy\}$$
$$=(x^2-y^2)(x^4+x^2y^2+y^4);$$

which latter expression is the same result as would be obtained by using a^3-b^3 instead of a^2-b^2, as above. The student may employ the reverse process, and thus make a good example for himself; that is, he may resolve x^6-y^6 by applying a^3-b^3, and convert it into the same form as is obtained by using a^2-b^2, viz., into the factors $(x^3+y^3)(x^3-y^3)$.

35. It may be noticed, in passing, that *elementary integral* expressions resembling, *in form*, a product of factors, can be resolved by the use of fractional indices. Thus,

$$x-y=(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x^{\frac{1}{2}}-y^{\frac{1}{2}}),$$

by imitating the factors of x^2-y^2, or

$$x-y=(x^{\frac{1}{3}}-y^{\frac{1}{3}})(x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}),$$

by imitating the factors of x^3-y^3.

36. As a specimen of the reasoning that may not unfrequently be applied to Algebraic expressions, the following deductions from the two preceding identities will neither be uninteresting nor useless. Suppose it is required to resolve a^7+b^7 into elementary factors: we proceed thus. The index is not the cube of any integral number, and therefore a^7+b^7 cannot be resolved simply by observing the above process. But 7 is an *odd* number, and so is 3; therefore there *is* a similarity between them. This being so, we presume from a notice of the two factors, $(a+b)(a^2-ab+b^2)$, that the binomial factor of a^7+b^7 will have unity for the index of each term, and that the resolution will therefore be

$$a^7+b^7=(a+b)\times \&c.$$

Thus far all will be clear. Now then for the next and more important step. It will be noticed from the other factor,

1*st, that a begins with the index next below that belonging to the a in the original product.*

2*nd, that the index of a gradually descends.*

3*rd, that the index of b is precisely the reverse.*

4*th, that the signs are alternate.*

$\therefore\ a^7+b^7=(a+b)(a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6)$.

So, again, in resolving $a^{11}-b^{11}$, the factors of a^3-b^3 must be imitated; and it will be seen that

$a^{11}-b^{11}=(a-b)(a^{10}+a^9b+a^8b^2+a^7b^3+a^6b^4+a^5b^5+a^4b^6+a^3b^7+a^2b^8+ab^9+b^{10})$.

These examples may be tested by ordinary multiplication or division, and the rule applied to any similar expressions.

36 (*a*). From what has been said and done, it will be seen

(*a.*) That when the index is *even* and sign *minus*, $a^n - x^n$ is divisible by both $a - x$ and $a + x$.

(*b.*) When the index is *even* and sign *plus*, $a^n + x$ is divisible by neither.

(*c.*) When the index is *odd* and sign *minus*, $a^n - x^n$ is divisible by $a - x$.

(*d.*) When the index is *odd* and sign *plus*, $a^n + x^n$ is divisible by $a + x$.

CHAPTER V.

The Cube of a Binomial.—Scheme for remembrance of Formulæ.—Extended application of the Cube of a Binomial.—Fully-worked illustrative Examples.—The form of a perfect Cube.—Examples on Chapters IV and V.

87.

$$\begin{array}{ll}
a+b & a-b \\
a+b & a-b \\
\hline
a^2+2ab+b^2 & a^2-2ab+b^2 \\
a+b & a-b \\
\hline
a^3+2a^2b+ab^2 & a^3-2a^2b+ab^2 \\
\quad +a^2b+2ab^2+b^3 & \quad -a^2b+2ab^2-b^3 \\
\hline
a^3+3a^2b+3ab^2+b^3 & a^3-3a^2b+3ab^2-b^3 \\
\hline\hline
\end{array}$$

The greater the number of formulæ brought under notice, the less will be the need of an exact and rigid analysis of each. The student will, doubtless, have already gained sufficient experience from the application of the foregoing, to be able, without assistance, to use the two above expansions for the purpose of finding the cube of any binomial. It will be a simple matter to remember the order in which the indices occur, and where and what the numeral coefficients are. However, we will just say—

1*st. That* a *gradually descends from the index of the expansion required.*

2*nd. That* b *gradually ascends to that power.*

3*rd. That each of the two middle terms has for its numerical coefficient the same number as forms the index above mentioned.*

Thus, then,

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3;$$
$$(x-y)^3 = x^3-3x^2y+3xy^2-y^3.$$

38. The student should here ask himself why the terms are alternately plus and minus in the second expansion. On careful consideration he will find this easily accounted for, by the fact that the sign of each term will, of course, depend on the *power of* $-b$ in it. Now in the second term it is simply the first power of $-b$, and therefore $(-b)^1$, which equals $-b$. In the third term it is the 2nd power of $-b$, or $(-b)^2$, and therefore $+b^2$. The fourth term has $-b$ to the 3rd power, or $(-b)^3$, and therefore $-b^3$. It will long ere this have been noticed that all *odd* powers of minus quantities have minus signs, while all *even* powers have plus signs, according to the rule that *like signs give plus in multiplication, and unlike minus.* For example,

$$(-b)^7 = (-b)^2 \times (-b)^2 \times (-b)^2 \times (-b)^1$$
$$= b^2 \times b^2 \times b^2 \times (-b) = b^6 \times (-b) = -b^7.$$

39. The extension of these formulæ to more important examples than those given above, will much depend, as with the others before, on the use of brackets. Two or three expansions will make their application clear.

(1.) $(p+q+x)^3$

$= \{(p+q)+x\}^3$

$= (p+q)^3+3(p+q)^2x+3(p+q)x^2+x^3$

$= p^3+3p^2q+3pq^2+q^3+3(p^2+2pq+q^2)x$
$+3px^2+3qx^2+x^3$

$= p^3+3p^2q+3pq^2+q^3+3p^2x+6pqx+3q^2x+3px^2+$
$3qx^2+x^3.$

(2.) $(m+n-y+z)^3$

$= \{(m+n)-(y-z)\}^3$

$= (m+n)^3 - 3(m+n)^2(y-z)+3(m+n)(y-z)^2 -$
$(y-z)^3$

$= m^3+3m^2n+3mn^2+n^3-3(m^2+2mn+n^2)(y-z)$
$+3(m+n)(y^2-2yz+z^2)-y^3+3y^2z-3yz^2+z^3$

$= m^3+3m^2n+3mn^2+n^3-3m^2y+3m^2z-6mny+6mnz$

$$-3n^2y+3n^2z+3my^2-6myz+3mz^2+3ny^2-6nyz$$
$$+3nz^2-y^3+3y^2z-3yz^2+z^3.$$

40. The above expansions have been left as they naturally formed themselves from the simple application of the identity at the commencement of the chapter, but they are capable of another and more symmetrical arrangement. For instance, the expansion of $(p+q+x)^3$

$$=p^3+q^3+x^3+3p^2(q+x)+3q^2(p+x)$$
$$+3x^2(p+q)+6pqx.$$

This form should be noticed by the student, but it is not so urgently pressed upon his attention as the foregoing, since it is comparatively seldom that trinomials and multinomials have to be cubed, and when they have, the binomial arrangement is, perhaps, the more easily remembered and applied.

41. A few examples will now be added, which must be carefully examined and clearly understood.

Show that

$$(p-q)^3+q^3-p^3=3pq(q-p).$$

This identity may be proved in the two following ways, the second of which deserves special attention, and will be hereafter referred to.

$$\begin{aligned}
&(a)\ (p-q)^3+q^3-p^3\\
&=p^3-3p^2q+3pq^2-q^3+q^3-p^3\\
&=3pq^2-3p^2q=3pq(q-p).
\end{aligned}$$

$$\begin{aligned}
&(b)\ (p-q)^3+q^3-p^3\\
&=(p-q)(p-q)^2-(p^3-q^3)\\
&=(p-q)(p^2-2pq+q^2)-(p-q)(p^2+pq+q^2)\\
&=(p-q)(p^2-2pq+q^2-p^2-pq-q^2)\\
&=-3pq(p-q)\\
&=3pq(q-p).
\end{aligned}$$

42. The use of the formulæ is, in every respect, such a matter for consideration, that the really thoughtful student will not always be content with the application of the one that may the most easily suggest itself, but will endeavour to combine economy of time with the true science of his subject. For instance, in multiplying

$$(a+b)^2 \text{ by } (a-b)^3,$$

the beginner may apply the processes for squaring and cubing a binomial, and then find the product of the two results. This method of solution would involve the use of two formulæ, but it would be a very inferior application. Instead of so doing, it must be noticed that

$$\begin{aligned} &(a+b)^2(a-b)^3 \\ &=(a+b)^2(a-b)^2(a-b) \\ &=(a^2-b^2)^2(a-b), \end{aligned}$$

which latter form the student will be able easily to manipulate for himself. Thus it will be clearly seen that the ability to discover the resemblances which suggest the use of an identity, is inferior to the judgement which selects the most appropriate. This latter is only to be gained by practice.

43. The form of a perfect cube should be carefully borne in mind, since it may often lead to the omission of useless work, as the following example will show. Thus, in simplifying

$$(a-b)^3+(a+b)^3+3(a-b)(a+b)^2+3(a-b)^2(a+b),$$

of course it is easy enough to work it out in a straightforward and partially mechanical way, and the true simplification will be obtained. It will, however, be seen, on careful consideration, that there is

1st. The cube of a first ....$(a-b)^3$;

2nd. The cube of a second ....$(a+b)^3$;

3rd. Three times a first squared into a second .... $3(a-b)^2(a+b)$;

4th. Three times a first into a second squared .... $3(a-b)(a+b)^2$

What then do these remarks suggest? Why simply that

$$(a-b)^3+(a+b)^3+3(a-b)(a+b)^2+3(a-b)^2(a+b)$$

resembles $x^3+3x^2y+3xy^2+y^3$,
but instead of x we get $a-b$, and instead of y we get $a+b$. Therefore the above expression

$$=\{(a-b)+(a+b)\}^3=\{a-b+a+b\}^3=(2a)^3.$$

This example must on no account be lightly passed over: it is of very great importance.

44. The following will also illustrate the manner in which the knowledge of the form of the perfect cube may often shorten work. Thus,

$$\frac{x^3+3x^2y+3xy^2+y^3+z^3}{x+y+z}$$

$$=\frac{(x+y)^3+z^3}{x+y+z}$$

$$=\frac{(x+y+z)\{(x+y)^2-(x+y)z+z^2\}}{x+y+z}$$

$$=(x+y)^2-(x+y)z+z^2.$$

This will be found far less laborious, and is much more scientific, than dividing the numerator by the denominator, in the form of an ordinary long division sum.

45. The student will see that there is no assignable limit to the manner in which either of the formulæ may be applied, the amount of work they may save, or the number of examples that may be adduced to illustrate their use.

(1.) $3(a^3 - 3a^2b + 3ab^2 - b^3) \div (a^2 - b^2)(a - b)^2$

$$= \frac{3(a-b)^3}{(a-b)(a-b)^2(a+b)} = \frac{3}{a+b}.$$

(2.) Required the L.C.M. of

$(x^3 - y^3)(x^3 - 3x^2y + 3xy^2 - y^3)$; $(x^4 - y^4)(x^2 - 2xy + y^2)$
and $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$.

$(x^3 - y^3)(x^3 - 3x^2y + 3xy^2 - y^3)$
$= (x - y)(x^2 + xy + y^2)(x - y)^3 = (x - y)^4(x^2 + xy + y^2)$;
$(x^4 - y^4)(x^2 - 2xy + y^2) = (x^2 - y^2)(x^2 + y^2)(x - y)^2$
$= (x - y)^3(x + y)(x^2 + y^2)$;
$x^6 - 3x^4y^2 + 3x^2y^4 - y^6 = (x^2 - y^2)^3$
$= (x - y)^3(x + y)^3$.
$\therefore$ L.C.M. $= (x - y)^4(x + y)^3(x^2 + y^2)(x^2 + xy + y^2)$.

This may, of course, be arranged in different ways, and it will be a very good exercise for the student to convert it into the form of

$$(x^4 - y^4)(x^3 - y^3)(x^2 - y^2)^2.$$

45 (*a*). Subjoined are a few examples on Chapters IV and V.

1. Find the value of $m^8 - n^8 \div m + n$.
2. Resolve $a^{18} - b^9$ into elementary factors.
3. Simplify $a^{18} + b^9 \div a^4 - a^2b + b^2$.
4. Reduce

$$\frac{(p^{12} - 2p^6q^6 + q^{12})(p^{12} + 2p^6q^6 + q^{12})}{(p^6 - 2p^3q^3 + q^6)(p^6 + 2p^3q^3 + q^6)(p^4 + 2p^2q^2 + q^4)}$$

to lowest terms.

5. Show that $\dfrac{x^6 + 3x^4y^2 + 3x^2y^4 + y^6 + m^6}{m^2 + y^2 + x^2}$

$= m^4+(x^2+y^2-m^2)(x^2+y^2)$.

6. Find the G.C.M. of $a^4+3a^3b+3a^2b^2+ab^3$, a^4+ab^3, a^2+ab, and $a^5-2a^3b^2+ab^4$.

7. Find the value of $(x^{\frac{1}{3}}+y^{\frac{1}{3}})(x^{\frac{1}{3}}-y^{\frac{1}{3}})(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}})$ $(x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}})$.

Simplify the following fractions—

8. $\dfrac{m^6+3m^4n^2+3m^2n^4+n^6-a^6+3a^4x^2-3a^2x^4+x^6}{m^2+n^2-a^2+x^2}$;

9. $\dfrac{8m^3+x^3+3x^2y+3xy^2+y^3}{y+2m+x}$;

10. $\dfrac{p^{13}q^{13}+m^{13}n^{13}}{pq+mn}$.

Resolve the following expressions into elementary factors—

11. $729x^{21}+343q^{27}$;
12. $(x+y)^3-(x+y-z)^3$;
13. $2(x^3+x^2y+xy^2)-x^3+y^3$;
14. $(p-q)^3-(p+q)^3+3(p^2-q^2)(p+q)-3(p^2-q^2)$ $(p-q)$;
15. $k^3-l^3-k(k^2-l^2)+l(k-l)^2$;
16. $m^{18}-n^{12}$.

Find the value of

17. $(a+b+c)^6$;
18. $(a+2b+3c+4d)^3$
19. 17^3+13^3;
20. 19^3-14^3.

CHAPTER VI.

Resolution of Trinomials.—Analysis of methods of Resolution.—Fully-worked illustrative Examples.—The Collection of Factors.—Solution of important Examples.—Change of Form by multiplication by unity.—Further Solutions.—Examples on Chapter VI.

46. In the foregoing chapters will be found the most essential points that the beginner of Algebra must master. Let him go carefully through this little work, conquering as he goes, and, although there will be difficulties to overcome,—where would be the pleasure if there were *no* difficulties?—he will find, in the end, that he has amassed such a store of really scientific knowledge, that, virtually, he will be far ahead of him who has gone through more advanced parts, without what has been here given at his "fingers' ends." It is the very pith of the subject, the hinges on which the science hangs, and can by no means be neglected by one who wishes to get up his Algebra thoroughly.

47. In the following pages a few additional hints will be thrown out, which the student will do well to remember and apply. The advantage of knowing them is great, and the use of them may be made a touchstone for the discovery of either real or somewhat superficial knowledge.

48. With regard to resolution into elementary factors, concerning which much has already been said, it may be here remarked, that

$$a^2 - b^2,\ a^3 + b^3, \text{ and } a^3 - b^3,$$

are very important types of quantities capable of being resolved. But there is another resolution to be mentioned, about which nothing has yet been said; and

$$x^2+(a+b)x+ab = (x+a)(x+b)$$

is an identity representing the various expressions now to be considered.

The above trinomial will yield, on analysis, a rule which is applicable to all other quantities of the same form.

48 (a). It will be seen,

1*st.* *That the coefficient of the 2nd term of the trinomial is the* SUM *of the two last terms of the binomial factors.*

2*nd.* *That the last term of the trinomial is the* PRODUCT *of the same two terms.*

Applying this process to other expressions, we see that since

$$a^2b^2+abx^2+aby^2+x^2y^2$$
$$= a^2b^2+ab\ (x^2+y^2)+x^2y^2,$$

the above law is here also involved.

$$\therefore\ a^2b^2+ab\ (x^2+y^2)+x^2y^2$$
$$= (ab+x^2)(ab+y^2).$$

Again,

$$x^2+15x+36 = (x+3)(x+12)\,;$$

and the student may usefully exercise himself by resolving other examples, and forming other products that shall be capable of resolution.

49. Here, however, it will be necessary to introduce a somewhat different specimen of the same process; as, for instance,

$$x^2-15\,x+50 = (x-5)(x-10).$$

In this example the signs require a little observation. Since the sign of the 2nd term of the trinomial is minus, there must be, at least, *one* minus in the binomial factors, as two plus signs, on addition, would give no minus for second term. Thus far all will be clear, but what about the sign of the other binomial factor? We will turn to the sign of the third term of the trinomial; it is plus. Now we know that the third term of the trinomial is the *product* of the two second terms of the binomial factors, and that *one* sign of these is minus. What then? Why it follows, *of necessity*, that they must *both* be minus, since the product of a plus and a minus *cannot give a plus*. The learner must go through this again and again, until he has thoroughly mastered it. It requires a little care, but is very easy. Similarly,

(1.) $x^4 - 19x^2 + 18 = (x^2 - 18)(x^2 - 1)$
$= (x^2 - 18)(x - 1)(x + 1)$.

(2.) $a^2 - 13ab + 42b^2 = (a - 6b)(a - 7b)$;
and so on.

50. Again,

$$x^2 - 7x - 44 = (x - 11)(x + 4).$$

Here is another variation, but one just as easily manipulated as the preceding. The sign of the third term of the trinomial, being the sign of a product, shows that one term of the binomial factors must have a minus sign, and the other a plus; while the second term of the trinomial indicates that the minus quantity must be seven greater than the plus. Thus,

$$x^2 - 7x - 44 = x^2 + (4 - 11)x - 44,$$

according to the law deduced in 48 (*a*), viz., that the two second terms of factors *added together* make -7, and *multiplied* make -44.

51. The student must make the case of a trinomial *having a numerical coefficient in the first term*, the subject of special notice for himself, as he will find that the coefficients, which will appear in the first terms of the binomial factors, will also influence the second and third terms; and the law of sum and product, as given above, will, consequently, be greatly modified. Thus,

(1.) $12x^4+x^2y^2-y^4=(3x^2+y^2)(4x^2-y^2)$
$=(3x^2+y^2)(2x+y)(2x-y).$

(2.) $9a^2b^2-3ab^3 \quad 6b^4$
$=3b^2\,(3a^2-ab-2b^2)=3b^2(3a+2b)(a-b).$

52. A few examples will now be added to illustrate the usefulness of this important resolution.

(1.) Simplify $\dfrac{x^2+3x+2}{x^2+6x+5}$.

$$\frac{x^2+3x+2}{x^2+6x+5}=\frac{(x+2)(x+1)}{(x+5)(x+1)}=\frac{x+2}{x+5}.$$

(2.) $\dfrac{y^2-10y+21}{y^2+10y-39}=\dfrac{(y-3)(y-7)}{(y-3)(y+13)}=\dfrac{y-7}{y+13}$.

(3.) Find the L.C.M. of $x^2-(a+b)x+ab$; $x^2-(a+c)x+ac$; and $x^2-(b+c)x+bc$.

$x^2-(a+b)x+ab=(x-a)(x-b)$;
$x^2-(a+c)x+ac=(x-a)(x-c)$;
$x^2-(b+c)x+bc=(x-b)(x-c)$;
$\therefore$ L.C.M. $=(x-a)(x-b)(x-c)$.

(4.) Find the L.C.M. of
y^2+3y+2; y^2+4y+3 and y^2+5y+6.
$y^2+3y+2=(y+2)(y+1)$;
$y^2+4y+3=(y+3)(y+1)$;
$y^2+5y+6=(y+3)(y+2)$;
$\therefore$ L.C.M. $=(y+1)(y+2)(y+3)$.

(5.) Find the G.C.M. of
y^4+y^2-6 and y^4-3y^2+2.
$y^4+y^2-6=(y^2+3)(y^2-2)$;
$y^4-3y^2+2=(y^2-1)(y^2-2)$;
$\therefore$ G.C.M. $=y^2-2$.

(6.) Divide $(y^2-3y+2)(y-3)$ by y^2-5y+6.

$$\frac{(y^2-3y+2)(y-3)}{y^2-5y+6}=\frac{(y-1)(y-2)(y-3)}{(y-2)(y-3)}$$
$$=y-1.$$

(7.) Resolve $(x+y)^2+(a+b+c)(x+y)+(a+b)c$ into factors.

A substitution may here take place, which will simplify the expression. Thus,

$$\text{Let } (x+y)=m,$$
$$\text{and } (a+b)=n.$$

Then $(x+y)^2+(a+b+c)(x+y)+(a+b)c$
$$=m^2+(n+c)m+nc=(m+n)(m+c)$$
$$=(x+y+a+b)(x+y+c).$$

The substitution, in this case, only simplifies by reducing the size of the expression, since the careful student will easily see that $a+b+c$, the coefficient of the 2nd term, is the *sum* of the same quantities which have for their *product*, $(a+b)c$, the third term of the trinomial.

53. After the resolution of Algebraic quantities into elementary factors, it frequently happens that a single expression has the same term or terms entering as a factor into one or more quantities, as in 41 (b), which example will here again be noticed.

Thus,

$$(p-q)^3+q^3-p^3=(p-q)(p-q)^2-(p^3-q^3)$$
$$=(p-q)(p^2-2pq+q^2)-(p-q)(p^2+pq+q^2).$$

Here the factor, $p-q$, is contained twice in the given expression, and has to be multiplied by two different quantities, viz., by $(p^2-2pq+q^2)$ and $-(p^2+pq+q^2)$. Now the student will doubtless remember that in multiplying out of brackets, each term of the multiplicand is multiplied by each term of the multiplier. Therefore, each of the terms, p and $-q$, must be multiplied separately by $p^2, -2pq, q^2, -p^2, -pq, -q^2$. Such being the case, it is evidently a waste of time to write $p-q$ twice, and quite useless to have p^2 and $-p^2$, q^2 and $-q^2$ occurring, and $-2pq$ and $-pq$ in two different terms. What then shall we do to avoid this loss of time and space? The remedy is found in what is called the *collection of factors;* and so $(p-q)(p^2-2pq+q^2)-(p-q)(p^2+pq+q^2)$ is written thus,

$$(p-q)(p^2-2pq+q^2-p^2-pq-q^2),$$

which equals $-3pq\ (p-q)$; certainly a shorter and more convenient form than the original. In such a case as this, the beginner is often led to ask how a single multiplication by $p-q$, can be the same as multiplying by the *two* factors $p-q$. The careful consideration of the above will answer the question; and he will at once see that if a has to be multiplied by $b+c-d-e+f+g-h$, it is far more convenient to write one a, and say

$$a(b+c-d-e+f+g-h),$$

than either $ab+ac-ad-ae+af+ag-ah$,
or $a(b+c)-a(d+e)+a(f+g-h)$,
or $a(b+c-d)-a(e-f-g+h)$,
&c., &c.

54. It will only be necessary to subjoin a few examples to show the immense importance of this collection of factors, without which many very simple expressions would become intolerably long and tedious.

(1.) Show that $(x+y+z)^3-x^3-y^3-z^3$
$=3(x+y)(y+z)(x+z)$.

$$(x+y+z)^3-x^3-y^3-z^3=\{(x+y)+z\}^3-x^3-y^3-z^3$$
$$=x^3+3x^2y+3xy^2+y^3+3x^2z+6xyz+3y^2z+3xz^2$$
$$+3yz^2+z^3-x^3-y^3-z^3$$
$$=3(x^2y+xy^2+x^2z+2xyz+y^2z+xz^2+yz^2)$$
$$=3\{x(xy+xz+yz+z^2)+y(xy+xz+yz+z^2)\}$$
$$=3\{(x+y)(xy+xz+yz+z^2)\}$$
$$=3[(x+y)\{y(x+z)+z(x+z)\}]$$
$$=3(x+y)(y+z)(x+z).$$

This example has been thus worked out, for the purpose of giving prominence to the collection of factors. It is easily seen, at commencement, that the 3 is contained in all the terms, and in the *required* form; therefore its factors are collected. We next notice that $x+y$ is another factor; consequently, we try to take the x out of such terms as will leave exactly the same factors by which to multiply it, as will be left for the y, when this latter is taken out of certain other terms. It will also be observed that $2xyz$ has been divided into xyz and xyz, the x being taken out of one to leave the term yz, and the y out of the other to leave xz, both of which are required. This step detracts somewhat, perhaps, from the value of the foregoing method of solution, but would be, by no means, disallowed. There is no difficulty whatever in seeing that x and y must *each* be multiplied by $(xy+xz+yz+z^2)$, since the two binomials remaining in the *required form*, after $(x+y)$ has been removed, give exactly this expression; so that the resolution into elementary factors is really a very simple process. A careful study of the above exercise will render all the others comparatively easy.

54 (*a*). In the manipulation of Algebraic expressions, especially in the case of rather formidable examples, like the one now under notice, it is useful to bear in mind that it is desirable not to alter, *if possible*, the form of any part of an expression, *if that very same form occurs in the required result.* Judgement must be used in determining whether the alteration of form is necessary or unnecessary.*

In the above example, for instance, the $(x+y)$ has been completely changed, and then brought back to exactly the same form again; whereas it is quite possible to simplify the expression, and retain the $(x+y)$ as an unaltered factor. Thus,

---

* See solution and "N.B." on page 11.

$$\begin{aligned}
&(x+y+z)^3 - x^3 - y^3 - z^3 = \{(x+y+z)^3 - z^3\} - \\
&\qquad (x^3+y^3) \\
&= (x+y+z - z)(x^2+y^2+z^2+2xy+2xz+2yz+xz+yz+ \\
&\qquad z^2+z^2)
\end{aligned}$$
[*]
$$\begin{aligned}
&\qquad - (x+y)(x^2 - xy+y^2) \\
&= (x+y)(x^2+y^2+3z^2+2xy+3xz+3yz - x^2+xy - y^2) \\
&= (x+y)(3z^2+3xy+3xz+3yz) \\
&= 3(x+y)(z^2+xy + xz+yz) \\
&= 3(x+y)\{y(x+z) + z(x+z)\} \\
&= 3(x+y)(y+z)(x+z).
\end{aligned}$$

54 (*b*). Further examples on the collection of factors will now be appended.

(2.) Show that

$$\begin{aligned}
&(x-y)^3+(y-z)^3+(z-x)^3 = 3(z-x)(y-z) \\
&\qquad (x-y). \\
&(x-y)^3+(y-z)^3+(z-x)^3 \\
&= \{(x-y)^3+(y-z)^3\}+(z-x)^3 \\
&= (x-y+y-z)(x^2 - 2xy+y^2 - xy+xz+y^2 - yz+y^2 \\
&\qquad - 2yz+z^2) + (z-x)(z-x)^2 \\
&= -(z-x)(x^2 - 3xy+3y^2+xz - 3yz+z^2)+(z-x) \\
&\qquad (z^2 - 2xz+x^2) \\
&= (z-x)(-x^2+3xy - 3y^2 - xz+3yz - z^2+z^2 - 2xz \\
&\qquad +x^2) \\
&= (z-x)(3xy - 3y^2 - 3xz+3yz) \\
&= 3(z-x)(xy - y^2 - xz+yz) \\
&= 3(z-x)\{y(x-y) - z(x-y)\} \\
&= 3(z-x)(y-z)(x-y).
\end{aligned}$$

This identity may be proved in other ways, and the work of this method may be shortened, but the collection of factors has been made the prominent point, and, therefore, it has been solved so as to ensure the success of the beginner, by proceeding gently step by step. This plan will always be found, at first, to be the safest; large steps often cause little blunders. It should be particularly noticed, how easily $(x-y)^3+(y-z)^3$ has been solved by using the factors of a^3+b^3, since $x-y$ stands for a, and $y-z$ for b. The $+(x-z)$, of the 3rd line, has been changed into $-(z-x)$, in the 4th, so as to make the $(z-x)$ common, and thereby enable its factors to be collected, the required form containing $z-x$.

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[*] By imitating the factors of $a^3 - b^3$; $(x+y+z)$ standing in the place of a, and z in the place of b.

(3.) $y(x^3+z^3)+xz(x^2-z^2)+z^3(x+z) \div (z+y)(x+z)$

$$= \frac{y(x+z)(x^2-xz+z^2)+xz(x-z)(x+z)+z^3(x+z)}{(z+y)(x+z)}$$

$$= \frac{(x+z)\{y(x^2-xz+z^2)+z(x^2-xz+z^2)\}}{(z+y)(x+z)}$$

$$= \frac{(z+y)(x^2-xz+z^2)}{z+y} = x^2-xz+z^2.$$

(4.) Show that

$(x+2y)x^3-(y+2x)y^3=(x-y)(x+y)^3.$

$(x+2y)x^3-(y+2x)y^3=x^4+2x^3y-y^4-2xy^3$

$=(x^4-y^4)+2xy(x^2-y^2)$

$=(x^2-y^2)(x^2+y^2)+2xy(x^2-y^2)$

$=(x^2-y^2)(x^2+2xy+y^2)=(x-y)(x+y)(x+y)^2$

$=(x-y)(x+y)^3.$

(5.) $\dfrac{3x^3-3x^2y+xy^2-y^3}{4x^2y-5xy^2+y^3}=\dfrac{3x^2(x-y)+y^2(x-y)}{y(4x^2-5xy+y^2)}$

$$=\frac{(3x^2+y^2)(x-y)}{y(x-y)(4x-y)}=\frac{3x^2+y^2}{y(4x-y}.$$

(6.) $\dfrac{y^2+9y+20}{y^3+7y^2+14y+8}=\dfrac{(y+5)(y+4)}{y^3+2^3+7y(y+2)}$

$$= \frac{(y+5)(y+4)}{(y+2)(y^2-2y+4)+7y(y+2)} = \frac{(y+5)(y+4)}{(y+2)(y^2+5y+4)}$$

$$= \frac{(y+5)(y+4)}{(y+2)(y+1)(y+4)} = \frac{y+5}{(y+2)(y+1)}.$$

(7.) $\left(\frac{a^3}{b^3}+\frac{b^3}{a^3}-\frac{a^2}{b^2}+\frac{b^2}{a^2}+\frac{a}{b}+\frac{b}{a}\right) \div \frac{a}{b}+\frac{b}{a}$

$$= \frac{\left(\frac{a^3}{b^3}+\frac{b^3}{a^3}\right)-\left(\frac{a^2}{b^2}-\frac{b^2}{a^2}\right)+\left(\frac{a}{b}+\frac{b}{a}\right)}{\frac{a}{b}+\frac{b}{a}}$$

$$= \frac{\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{a^2}{b^2}-1+\frac{b^2}{a^2}\right)-\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{b}+\frac{b}{a}\right)}{\frac{a}{b}+\frac{b}{a}}$$

$$= \frac{\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{a^2}{b^2}-1+\frac{b^2}{a^2}-\frac{a}{b}+\frac{b}{a}+1\right)}{\frac{a}{b}+\frac{b}{a}}$$

$$= \frac{a^2}{b^2}+\frac{b^2}{a^2}-\frac{a}{b}+\frac{b}{a}.$$

(8.) Show that $p(p-2q)^3 - q(q-2p)^3 = (p-q)(p+q)^3$.

$$p(p-2q)^3 - q(q-2p)^3$$
$$= p(p^3-6p^2q+12pq^2-8q^3)-q(q^3-6pq^2+12p^2q-8p^3)$$
$$= p^4-6p^3q+12p^2q^2-8pq^3-q^4+6pq^3-12p^2q^2+8p^3q$$
$$= p^4+2p^3q-2pq^3-q^4$$
$$= (p^4-q^4)+2pq(p^2-q^2)$$
$$= (p^2-q^2)(p^2+q^2)+2pq(p^2-q^2)$$
$$= (p^2-q^2)(p^2+q^2+2pq)$$
$$= (p-q)(p+q)(p+q)^2 = (p-q)(p+q)^3.$$

55. If careful and intelligent attention he given to the foregoing methods and remarks, there will be little else that the learner needs to trouble himself about, in order to thoroughly understand and enjoy his subject. But he must be continually on the look-out for the introduction of any little manœuvre that may shorten labour, and, contrary to general rules, *must take rather a delight in scheming.* This is no where more clearly shown than in some few exercises that will follow; but the artifice employed therein is only a specimen of others that may be applied in different cases.

(1.) Find the value of $\frac{x}{2x-2y}+\frac{y}{2y-2x}$

In addition of fractions, it is, of course, necessary to find the least common denominator; and, at first sight, it appears evident that the one for this example is

$$(2x-2y)(2y-2x);$$

but it requires not much penetration to see that, if by any means $2y-2x$ can be changed into $2x-2y$, the L.C.D. will be *one* of the quantities instead of *both*; and this very much to the advantage of the solution. Now, it is well known, that if the numerator and denominator of a fraction be both multiplied by the same number, the value of the fraction is not altered, since the process is equal to a multiplication by unity.

$$\text{Thus, } \frac{+y}{2y-2x}\times\frac{-1}{-1}=\frac{-y}{-2y+2x}=\frac{-y}{2x-2y}.$$

$$\therefore \frac{x}{2x-2y}+\frac{y}{2y-2x}=\frac{x}{2x-2y}-\frac{y}{2x-2y}$$

$$=\frac{x-y}{2x-2y}=\frac{x-y}{2(x-y)}=\frac{1}{2}.$$

This change of form will admit of other explanations, but of none shorter or clearer than the above.

$$(2.)\ \frac{3+2y}{2-y}-\frac{2-3y}{2+y}+\frac{16y-y^2}{y^2-4}$$
$$=\frac{3+2y}{2-y}-\frac{2-3y}{2+y}+\frac{y^2-16y}{4-y^2}$$
$$=\frac{(2+y)(3+2y)-(2-y)(2-3y)+y^2-16y}{4-y^2}$$
$$=\frac{6+7y+2y^2-4+8y-3y^2+y^2-16y}{4-y^2}$$
$$=\frac{2-y}{(2-y)(2+y)}=\frac{1}{2+y}.$$

(3.) Simplify $\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}$
$$+\frac{1}{(c-a)(c-b)}.$$

As the factors stand here, the L.C.D. will be
$$(a-b)(a-c)(b-a)(b-c)(c-a)(c-b);$$
but $(a-b)$ and $(b-a)$, $(a-c)$ and $(c-a)$, $(b-c)$ and $(c-b)$, are factors differing only in sign; therefore if we change the signs of one of each pair by the former method, *three* of the factors will be the L.C.D. instead of all the six, since

$\frac{+1}{(b-a)(b-c)}$ becomes $\frac{-1}{(a-b)(b-c)}$, and $\frac{+1}{(c-a)(c-b)}$ becomes $\frac{+1}{(a-c)(b-c)}$. This needs some explanation, which the student must clearly understand. The first may be seen without difficulty to change thus,

$$\frac{+1}{(b-a)(b-c)}=\frac{1}{b-a}\times\frac{1}{b-c}=\frac{1}{b-a}\times\frac{-1}{-1}\times\frac{1}{b-c},$$

$$= \frac{-1}{-b+a} \times \frac{1}{b-c} = \frac{-1}{(a-b)(b-c)};$$

but the second has two factors to change, and, therefore, each must be multiplied by the changing quantity, as follows:—

$$\frac{+1}{(c-a)(c-b)} = \frac{1}{c-a} \times \frac{1}{c-b} = \frac{1}{c-a} \times \frac{-1}{-1} \times \frac{1}{c-b} \times \frac{-1}{-1}$$

$$= \frac{-1}{-c+a} \times \frac{-1}{-c+b} = \frac{+1}{(a-c)(b-c)}.$$

$$\therefore \frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}$$

$$= \frac{1}{(a-b)(a-c)} - \frac{1}{(a-b)(b-c)} + \frac{1}{(a-c)(b-c)}$$

$$= \frac{b-c-a+c+a-b}{(a-b)(a-c))b-c)} = 0.$$

(4.) $$\frac{1}{x(x-y)(x-z)} + \frac{1}{y(y-z)(y-x)} + \frac{1}{z(z-x)(z-y)}$$

$$= \frac{1}{x(x-y)(x-z)} - \frac{1}{y(y-z)(x-y)} + \frac{1}{z(x-z)(y-z)}$$

$$= \frac{yz(y-z) - xz(x-z) + xy(x-y)}{xyz(x-y)(x-z)(y-z)}$$

$$= \frac{y(yz - z^2 + x^2 - xy) - xz(x-z)}{xyz(x-y)(x-z)(y-z)}$$

$$= \frac{y\{(x-z)(x+z) - y(x-z)\} - xz(x-z)}{xyz(x-y)(x-z)(y-z)}$$

$$= \frac{y(x-z)(x+z-y) - xz(x-z)}{xyz(x-y)(x-z)(y-z)}$$

$$=\frac{(x-z)(xy+yz-y^2-xz)}{xyz(x-y)(x-z)(y-z)}$$

$$=\frac{x(y-z)-y(y-z)}{xyz(x-y)(y-z)}$$

$$=\frac{(x-y)(y-z)}{xyz(x-y)(y-z)}=\frac{1}{xyz}.$$

The above example is worthy of much study; indeed, it must not be passed over until it is seen that each step is both clear and easy. The following solution demands similar attention, and is quite as simple and scientific.

$$\frac{1}{x(x-y)(x-z)}+\frac{1}{y(y-z)(y-x)}+\frac{1}{z(z-x)(z-y)}$$

$$=\frac{1}{x(x-y)(x-z)}-\frac{1}{y(y-z)(x-y)}+\frac{1}{z(x-z)(y-z)}$$

$$=\frac{yz(y-z)-xz(x-z)+xy(x-y)}{xyz(x-y)(x-z)(y-z)}$$

$$=\frac{yz(y-z)-x^2z+xz^2+x^2y-xy^2}{xyz(x-y)(x-z)(y-z)}$$

$$=\frac{yz(y-z)+x^2(y-z)-x(y^2-z^2)}{xyz(x-y)(x-z)(y-z)}$$

$$=\frac{yz(y-z)+x^2(y-z)-x(y+z)(y-z)}{xyz(x-y)(x-z)(y-z)}$$

$$=\frac{(y-z)(yz+x^2-xy-xz)}{xyz(x-y)(x-z)(y-z)}$$

$$=\frac{x(x-z)-y(x-z)}{xyz(x-y)(x-z)}=\frac{(x-y)(x-z)}{xyz(x-y)(x-z)}$$

$$\frac{1}{xyz}.$$

(5.) Show that

$$(a-b)(x-a)(x-b)+(b-c)(x-b)(x-c)+(c-a)(x-c)(x-a)=(a-c)(a-b)(b-c).$$

$$
\begin{aligned}
&(a-b)(x-a)(x-b)+(b-c)(x-b)(x-c)+(c-a)(x-c)(x-a)\\
=\;&(x-b)(ax-a^2-bx+ab+bx-bc-cx+c^2)+(c-a)(x-c)(x-a)\\
=\;&(x-b)(ax-a^2+ab-bc-cx+c^2)+(c-a)(x-c)(x-a)\\
=\;&(x-b)\{(c+a)(c-a)-b(c-a)-x(c-a)\}+(c-a)(x-c)(x-a)\\
=\;&(x-b)(c+a-b-x)(c-a)+(c-a)(x-c)(x-a)\\
=\;&(c-a)(cx+ax-bx-x^2-bc-ab+b^2+bx)+(c-a)(x-c)(x-a)\\
=\;&(c-a)(cx+ax-x^2-bc-ab+b^2+x^2-ax-cx+ac)\\
=\;&(c-a)(-bc-ab+b^2+ac)\\
=\;&(a-c)(bc+ab-b^2-ac)\\
=\;&(a-c)\{a(b-c)-b(b-c)\}\\
=\;&(a-c)(a-b)(b-c).
\end{aligned}
$$

Enough has been said and done to show the principle and value of the collection of factors. Each example given is a complete study for the beginner; and, would he mount with ease to the height of success in this important and interesting work, he must, on no account, disregard the little steps by which alone he will be able to secure for himself a footing both safe and permanent.

56. Subjoined are a few examples which embrace the substance, and illustrate the utility of Chapter VI.

Resolve the following expressions into elementary factors :—

1. $m^2+29mn+210n^2$.
2. $x^2y^2+37xyz+342z^2$.
3. $p^4q^4-21p^2q^2r+104r^2$.
4. $a^8z^2-19a^4p^2z+18p^4$.
5. $a^8z^2-19a^4p^2z-20p^4$.
6. $b^6y^4z^2+11b^3y^2q^2z-242q^4$.
7. $a^2+(m+n)a+mn$.
8. $(p+q)^2+13(px+py+qx+qy)+42(x+y)^2$.
9. $3a^2m^2-42amq+120q^2$.
10. $2(m+n)^2-7(m+n)(x+y)+3(x+y)^2$.
11. $p^2q^2-130pqxy+129x^2y^2$.

12*. $x-3-(3-x)(x+1)=x(x-3)+8$.

13. $\frac{x}{3}-\frac{1}{3}-\frac{x}{4}+\frac{1}{4}=\frac{x}{5}-\frac{1}{5}-\frac{x}{6}+\frac{1}{6}$.

Find the value of

14. $\frac{x^2+13x+36}{x^2+16x+63}$.

15. $\frac{1}{p^2-(a+b)p+ab}+\frac{1}{p^2-(a+c)p+ac}+\frac{1}{p^2-(b+c)p+bc}$

16. $\frac{12m^4+m^2n^2-n^4}{9m^4+6m^2n^2+n^4}$

---

* Be careful to COLLECT FACTORS whenever the process will be of advantage.

17. $\frac{1}{(x-y)\ (x-z)}+\frac{1}{(y-x)\ (y-z)}+\frac{1}{(z-x)\ (z-y)}.$

18. $\frac{p^2-pm-pn+mn}{p^2-pq-pm+mq}$

19. $\frac{2a^2x^2+5axy+2y^2}{6a^2x^2+7axy+2y^2}$

20. $\frac{p(p-2y)^3-y(y-2p)^3}{p^3+3p^2y+3py^2+y^3}$

21. $\frac{(x+y)^2-11c\ (x+y)+30c^2}{x-5c+y}$

Change the left-hand member of each of the following six examples into the form of the right-hand member :—

22. $(p-q)^3+(q-m)^3+(m-p)^3=3(m-p)\ (q-m)\ (p-q).$

23. $(p+q+r)^3-p^3-q^3-r^3=3\ (p+q)\ (q+r)\ (p+r).$

24. $(a-b)\ (x-a)\ (x-b)+(b-c)\ (x-b)\ (x-c)$
$+\ (c-a)\ (x-c)\ (x-a)=(a-b)\ (b-c)\ (a-c).$

25. $(x+y+z)^3-(y+z-x)^3-(x-y+z)^3-x+y-z)^3$
$=24xyz.$

26. $\frac{3+2x}{2-x}-\frac{2-3x}{2+x}+\frac{16x-x^2}{x^2-4}+\frac{1}{x+2}.$

27. $\frac{\frac{x^3}{y^3}+\frac{y^3}{x^3}-\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{x}{y}+\frac{y}{x}}{\frac{x}{y}+\frac{y}{x}}=\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{x}{y}+\frac{y}{x}.$

Find the L.C.M. of

28. $x^2-13x+42$, x^2-9x+4, and $x^2-12x+35$.

29. x^4-144, $x^4-25x+156$, and $x^4-5x-84$.

80. p^6+20p^3+19, p^6-361, p^5+19p^2, and p^6q+18p^3q-19q.

81.[*] $4k^2-9y^2$, $4k^2-y^2$, $4k^2-8ky+3y^2$, $4k^2+4ky-3y^2$. $4k^2-4ky-3y^2$, and $4k^2+8ky+3y^2$.

Find the G.C.M. of

82. x^2-4x+4, and $x^2-7x+10$.

83. $2x^4-288$, and $x^4y^2-11x^2y^2-12y^2$.

84. $p^2+21pq+110q^2$, $p^2+pq-90q^2$, and $2p^2-2pq-220q^2$.

[*] Science Examination, May 1870.

CHAPTER VII.

Important Formula for high Expansions.—Application of formula.—Examples.—Useful Formula for Multiplication.—Fully-worked illustrative Examples.—Remarks on Manipulation.—Hints on symmetrical arrangement.—Remarks and Examples on Division.—Observations on correct study.—Important Mathematical Theorem.—Illustrative Examples.

57. In addition to the formulæ already given, and so urgently pressed upon the student's attention, without a command of which an Algebraist can have but a superficial knowledge of his subject, various important points require notice; and to these some little space will now be devoted. One or two formulæ will also be mentioned, which, although of less value than the foregoing, nevertheless require careful and studious consideration.

58. It is not unfrequently the case, that the learner finds some little difficulty in expanding an Algebraic expression, beyond a certain low power to which he has been accustomed. Of course a high expansion may be accomplished by various combinations of the lower powers; thus,

$$(a+b)^4 = \{(a+b)^2\}^2,$$
$$(a+b)^5 = (a+b)^2(a+b)^3,$$
$$(a+b)^6 = \{(a+b)^3\}^2,$$

and so on; but the process is frequently long and tedious. The difficulty, however, may be altogether removed, by the application of the following formula, viz., that

$$(x+y)^m = x^m + \frac{m}{1}x^{m-1}y + \frac{m(m-1)}{1\times2}x^{m-2}y^2$$
$$+ \frac{m(m-1)(m-2)}{1\times2\times3}x^{m-3}y^3 + — — — — + y^m,$$

the *form* of which alone contains any element even of *apparent* difficulty. Let us apply it, and find the expansion of $(a+b)^6$. Here, then, $m = 6$, and thus we have, by a close imitation of the above,—

$$(a+b)^6 = a^6+\frac{6}{1}a^5b+\frac{6\times5}{1\times2}a^4b^2+\frac{6\times5\times4}{1\times2\times3}a^3b^3$$

$$+\frac{6\times5\times4\times3}{1\times2\times3\times4}a^2b^4+\frac{6\times5\times4\times3\times2}{1\times2\times3\times4\times5}ab^5+b^6$$

$$= a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6.$$

In expanding $(a-b)^6$, every term containing an odd power of $-b$, must evidently have a minus sign, (Art. 38); and, therefore,

$$(a-b)^6 = a^6-6a^5b+15a^4b^2-20\,a^3b^3+15a^2b^4$$
$$-6ab^5+b^6.$$

Again, $(2x+3y)^4$

$$= (2x)^4+4(2x)^3(3y)+\frac{4\times3}{1\times2}(2x)^2(3y)^2+\frac{4\times3\times2}{1\times2\times3}(2x)$$
$$(3y)^3+(3y)^4$$

$$= 16x^4+96x^3y+216x^2y^2+216xy^3+81y^4.$$

Also, $(a+b+c)^5 = \{(a+b)+c\}^5$

$$= (a+b)^5+5(a+b)^4c+\frac{5\times4}{1\times2}(a+b)^3c^2+\frac{5\times4\times3}{1\times2\times3}(a+b)^2$$
$$c^3$$
$$+\frac{5\times4\times3\times2}{1\times2\times3\times4}(a+b)c^4+c^5$$

$$= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$
$$+5c(a^4+4a^3b+6a^2b^2+4ab^3+b^4)+10c^2(a^3+3a^2b+$$
$$3ab^2+b^3)$$
$$+10c^3(a^2+2ab+b^2)+5c^4(a+b)+c^5,$$

which, for simple and entire completion, needs now only to be taken out of brackets.

59. The above illustrations will be quite sufficient to impress upon the mind of the pupil the desirability of understanding and applying the symmetrical arrangement contained in the preceding identity; and the few following examples, for practice, will make the formula clearer and easier than many words.

1. $(a+y)^4$. 2. $(2a-y)^4$.
3. $(2x+3a)^5$. 4. $(a-4x)^5$.
5. $(x+y+z)^5$. 6. $(x-y+z)^6$.
7. $(a-b)^7$. 8. $(a-b)^8$.
9. $(4a+6b+1)^3$. 10. $(y+3)^7$.
11. $(m+n)^9$. 12. $(m-n)^{10}$.

60. Again, in Article 48, it has been brought before the student's notice, that

$$(x+a)(x+b) = x^2+(a+b)x+ab;$$

and it may be shown, by actual multiplication, that

$$(x+a)(x+b)(x+c) = x^3+(a+b+c)x^2+(ab+ac+bc)x+abc;$$

&c., &c. A high process of reasoning, called *mathematical induction*, shows that this symmetrical arrangement holds good for any number of binomial factors; and, therefore, it becomes a very useful formula for finding the product of a factorial expression, corresponding, *in form*, to the left-hand member of either of the above identities. The process is contained in the following examples :—

(1.) $(x+2y)(x+3y) = x^2+(2y+3y)x+6y^2 = x^2+5xy+6y^2$.

(2.) $(x+3y)(x+9y)(x+5y) = x^3+(3y+9y+5y)x^2+(27y^2+15y^2+45y^2)x+135y^3$

$$= x^3+17x^2y+87xy^2+135y^3.$$

(3.) $(4p-2q)(4p-6x)(4p+m)$

$$= 64p^3-(2q+6x-m)(4p)^2+(12qx-2qm-6mx)(4p)+12qmx$$

$$= 64p^3-16(2q+6x-m)p^2+8(6qx-qm-3mx)p+12qmx.$$

61. Every possible use must be made of this combination, and the student must avail himself of its assistance, in the solution of any examples into which it can be satisfactorily introduced. Of course, the complete mastery of a formula is shown only by an ability to apply it whenever it can possibly be admitted with ease and advantage. Thus,

(1.) $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc \div x^2+(a+b)x+ab$

$$= \frac{(x+a)(x+b)(x+c)}{(x+a)(x+b)} = x+c.$$

(2.) $\frac{1}{p^2+(m+n)p+mn}+\frac{1}{p^2+(m-n)p-mn}+\frac{1}{p^2+(m+q)p+mq}$

$$= \frac{1}{(p+m)(p+n)}+\frac{1}{(p+m)(p-n)}+\frac{1}{(p+m)(p+q)}$$

$$= \frac{(p+q)(p-n)+(p+q)(p+n)+(p+n)(p-n)}{(p+m)(p+n)(p-n)(p+q)}$$

$$= \frac{(p+q)(p-n+p+n)+p^2-n^2}{(p+m)(p+n)(p-n)(p+q)}$$

$$= \frac{2p(p+q)+p^2-n^2}{(p+m)(p+n)(p-n)(p+q)}.$$

62. Much notice must be taken of the *form* of

the foregoing identities, so that they may be firmly fixed on the mind; but, on careful examination, the pupil will find their symmetrical arrangement to be so easy and regular, that he will not experience the least difficulty in remembering and applying them. No hints to that effect need here be given.

63. Show that

$$(m^2+n^2)(p^2+q^2) = (mp+nq)^2+(mq-np)^2.$$

$$(m^2+n^2)(p^2+q^2) = m^2p^2+m^2q^2+n^2p^2+n^2q^2$$
$$= m^2p^2+m^2q^2+n^2p^2+n^2q^2+2mnpq-2mnpq$$
$$= (m^2p^2+2mnpq+n^2q^2)+(m^2q^2-2mnpq+n^2p^2)$$
$$= (mp+nq)^2+(mq-np)^2.$$

This last example includes a device, which the careful student must have noticed, when $+2mnpq$ and $-2mnpq$ were added to the expression. Now, it is evident that the value cannot be affected by such an alteration; but what is there to suggest it? Why the right-hand member of the identity, on being *mentally* squared, gives these two terms in addition to the ones obtained by the multiplication of the left-hand factors: and, therefore, they are required to complete the *form*, although they in no way affect the *value*. This is merely another specimen of the allowable and interesting schemes, which the ingenuity of the student must ever be on the alert to apply.

Of course it would have been quite sufficient for the establishment of the equality, if the right-hand member of the identity had been converted into the form of the left-hand factors, as below; but, in many cases, it is not possible to determine, at a glance, which is the better process for manipulation, and then it is more natural to commence at once with the side of the identity first given. Beginning with the right-hand member, we have

$$\begin{aligned}&(mp+nq)^2+(mq-np)^2\\ &=m^2p^2+2mnpq+n^2q^2+m^2q^2-2mnpq+n^2p^2\\ &=m^2(p^2+q^2)+n^2(p^2+q^2)\\ &=(m^2+n^2)(p^2+q^2).\end{aligned}$$

64. One very important observation will now be made, relative to the working of certain Algebraic examples, which contain terms having regularly ascending or descending powers of some common letter. The student should bear in mind that it is generally desirable, and, in some instances, *necessary*, to arrange the terms in the order of the indices of the common letter: at the least, it is awkward not to do so, and ill becomes a good Algebraist.

Thus,

$$x^4-15y^4-8x^2y^2+x^3y+19xy^3 \div 3xy-5y^2+x^2$$

is very badly arranged for division, and, as it stands, would be likely to defy the utmost endeavours of the learner to solve it. The dividend and divisor should imitate one another in the position of the terms; and, since it will be very convenient to arrange according to descending powers of x, the above example will nicely become

$$x^4+x^3y-8x^2y^2+19xy^3-15y^4 \div x^2+3xy-5y^2,$$

the solution of which will present no element of difficulty.

65. Not only is this the case, but, *still more important*, the first term, for each successive division, must be brought down also in the order of the indices of the common letter which has been selected to govern the arrangement. Thus, if x^4 be the first term in the dividend, and x the first in the divisor, the next line for division should commence with a term having in it x^3, the following x^2, and so on;

unless, indeed, the quantities, thus noticed, have disappeared in the ordinary working of the sum. The meaning of these remarks is far too evident to need illustration; experience will make their importance manifest.

66. It frequently happens, however, that there will be more than one term containing the power of the quantity which has to be brought down: thus, if x^3 be the letter to appear in the first term of the line, there may be two, or three, or even more terms having the x^3 in it. What then is to be done? Is the x^3 to appear for so many following divisions, in order to get rid of all the terms containing it, and thus proceed to the x^2? By no means. The tyro may do this, but the process would at once discover the standard of the operator. Let the factors of the x^3 be collected, and thus, by a kind of compound term in the quotient, the division for this power will be immediately accomplished, and the progress through the example be made short and sure. A few illustrations will remove all doubts and difficulties.

Fully-worked Illustrative Example.

(1.) $ab^3+2b^3c-ab^2c+abc^2-a^3b-2bc^3+a^3c-ac^3 \div b+c-a$

$= b^3(a+2c)-ab^2c+b(ac^2-a^3-2c^3)+a^3c-ac^3 \therefore b+c-a$

$$b+c-a \,\big]\, b^3(a+2c)-ab^2c+b(ac^2-a^3-2c^3)+a^3c-ac^3 \,\big[\, b^2(a+2c)+b(a^2-2c^2)-(a^2c+ac^2).$$

$$\underline{b^3(a+2c)+b^2c(a+2c)-ab^2(a+2c)}$$

$$\begin{cases} -2ab^2c-2b^2c^2+a^2b^2+2ab^2c \\ =b^2(a^2-2c^2) \end{cases}$$

$$\underline{b^2(a^2-2c^2)+bc(a^2-2c^2)-ab(a^2-2c^2)}$$

$$\begin{cases} -a^2bc-abc^2 \\ =-b(a^2c+ac^2) \end{cases}$$

$$\underline{-b(a^2c+ac^2)-c(a^2c+ac^2)+a(a^2c+ac^2)}$$

$$0 \qquad 0 \qquad 0$$

In this example the terms are very easily arranged in the order of the descending powers of b, the factors being collected where more than one term contains the same power of the letter. The student should very carefully work this through, and altogether satisfy himself concerning every term and step in the process. Where necessary, the terms of each subtrahend must be mentally taken out of brackets, and, when they do not cancel others, be set down in the remainder, in readiness for the next division. A careful analysis of the above work will impress the method more thoroughly on the memory than a page of explanation.

$$
\begin{array}{l}
(2.)\,{}^{*}a+b+c\,\big]\,a^3+3a^2b+3ab^2+b^3+c^3\,\big[\,a^2+a(2b-c)+b^2-bc+c^2 \\
\qquad\qquad\quad\; \underline{a^3+a^2b+a^2c} \\
\qquad\qquad\quad \left\{\begin{array}{l} 2a^2b-a^2c \\ =a^2(2b-c) \end{array}\right. \\
\qquad\qquad\quad\; \underline{a^2(2b-c)+ab(2b-c)+ac(2b-c)} \\
\qquad\qquad\quad \left\{\begin{array}{l} ab^2-abc+ac^2 \\ =a(b^2-bc+c^2) \end{array}\right. \\
\qquad\qquad\quad\; \underline{\underline{a(b^2-bc+c^2)+b(b^2-bc+c^2)+c(b^2-bc+c^2)}}
\end{array}
$$

---

* See Article 44, where a preferable method of Solution is adopted. This form is given here to illustrate the subject in question.

$$
\begin{array}{l}
(3.)\ y-1\Big]\,y^5-my^4+ny^3-ny^2+my-1\Big[\,y^4-y^3(m-1)-y^2(m-n-1)-y(m-1)+1 \\
\qquad\qquad \underline{y^5-y^4} \\
\qquad\qquad \left\{\begin{array}{l} -my^4+y^4 \\ -y^4(m-1) \end{array}\right. \\
\qquad\qquad \underline{-y^4(m-1)+y^3(m-1)} \\
\qquad\qquad \left\{\begin{array}{l} -y^3(m-1)+ny^3 \\ -y^3(m-n-1) \end{array}\right. \\
\qquad\qquad \underline{-y^3(m-n-1)+y^2(m-n-1)} \\
\qquad\qquad \left\{\begin{array}{l} -y^2(m-n-1)-ny^2 \\ -y^2(m-1) \end{array}\right. \\
\qquad\qquad \underline{-y^2(m-1)+y(m-1)} \\
\qquad\qquad \left\{\begin{array}{l} -y(m-1)+my-1 \\ \quad y-1 \end{array}\right. \\
\qquad\qquad \quad \underline{y-1}
\end{array}
$$

67. The following hint is not unworthy of notice, since it induces an insight into the nature of Algebraic expressions, and is calculated to promote freedom of manipulation. Suppose it is required to find the G.C.M. of

$$x^2 - 3x - 70,\ x^3 - 39x + 70, \text{ and } x^3 - 48x + 7.$$

Now $x^2 - 3x - 70 = (x - 10)(x + 7)$, and the student can see at once that $x - 10$ can, by no possibility, be contained in $x^3 - 48x + 7$, since 7 is no multiple whatever of 10.

$$\therefore \text{G.C.M.} = x + 7.$$

Numerous similar examples may be adduced, but a careful consideration of the above will be sufficient.

68. There is no part of a beginner's Algebra that is more systematically slighted, and this, too, not unfrequently, with the knowledge and consent of his teacher, than those pages of introduction which usually accompany the commencement of a chapter, and precede a set of examples. They are, to him, often nothing more than waste paper; and the only interest attached to them is that they serve, without trouble, to bring him so much nearer to his goal—the *Finis*. But these pages are the muscles of the subject—the exercises tend simply to develop them—and they cannot be disregarded without entailing puny growth, and weak and meagre progress. It is a mental training of the highest character, to read and understand them; it is a sure mark of failure to omit them. The *proofs of rules*, for instance, contained therein, need the most thorough mastery; without which, the working of the examples is simply a groping in the dark, where one may get along while the way is straight and smooth, but where no difficulty can be encountered with certainty, or guidance given to others.

69. As a specimen of the work referred to, the pupil should carefully study the following proposition, until its method and meaning are to him both clear and simple. It is a very important mathematical theorem, ever and absolutely true, and even for this reason alone is worthy of the minutest investigation. It may be stated as follows :—

Any measure of two quantities must also be a measure of the difference between them; and, consequently, must either be this difference itself, or a factor of it.

Thus, let A and B be two Algebraical expressions, of which B is the greater, the difference between them being x; and let m, their greatest common measure, be contained n times in A. Since it is the G.C.M., m must also, of course, be contained in B, that is, in $A+x$, or in $mn+x$. Now m goes evenly in $mn+x$, and also in mn; therefore m must be contained *evenly* in x, the difference between A and B; and it is evident that m cannot possibly be contained in x LESS THAN ONCE; therefore m cannot be greater than x. Thus the truth of the proposition is established.

69 (*a*). The few following examples will clearly and correctly illustrate the above theory.

(1.) Find the G.C.M. of

$x^3+16x^2-45x+228$, and $x^3+18x^2-51x+252$.

Now the difference between these two expressions

$=x^3+18x^2-51x+252-(x^3+16x^2-45x+228)$

$=2x^2-6x+24=2(x^2-3x+12)$;

and it is evident, from inspection, that 2 cannot possibly be the G.C.M., *or a factor of it*, since the coefficient of x^3, in each case, is unity, and will not therefore admit of integral division by 2.

Also, $x^2 - 3x+12$ is elementary, as the *sum* of no two numbers whose *product* is 12, can possibly equal 3.*

$\therefore$ G.C.M. $= x^2 - 3x+12$.

(2.) Find the G.C.M. of

x^3+8x^2-8x-1, and $x^3+32x^2+208x+23$.

$x^3+32x^2+208x+23-(x^3+8x^2-8x-1)$
$=24x^2+216x+24=24(x^2+9x+1)$.

$\therefore$ G.C.M. $=x^2+9x+1$.

(3.) Similarly, the G.C.M. of
$p^3-41p-30$, and $p^3-11p^2+25p+25=p^2-6p-5$.

(4.) Again,
$2m^3+49m^2+344m+585-(2m^3+44m^2+262m+364)$
$=5m^2+82m+221=(5m+17)(m+13)$.

$\therefore$ The G.C.M. of these two expressions $=m+13$.

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* See Article 48 (*a*).

CHAPTER VIII.

Remarks on easy Manipulation.—Exercises for easy Manipulation.—Conclusion.

70. The student, who has carefully gone through the various sections of this little work, must have been more than once impressed with the exceeding importance *of an easy manipulation of Algebraic expressions.* And he may rest assured that there is no faculty which will render him more assistance and pleasure in the prosecution of this entertaining study, than the ability to discover, without delay, the surest method of attack, and the shortest and safest plan for the application of this method. It will be far from a waste of time for him to endeavour to solve many of the problems that will come under his notice, in more ways than one; and it is not unfrequently the case, that, during a moment of leisure, when an example is being carefully turned over in the mind, an easy plan of solution will be suddenly hit upon, so that difficulties, which before seemed almost insurmountable, will at once vanish before this, perhaps, new application of a formula. It is the acquirement of this power of easy manipulation, which alone can materially and usefully shorten the beginner's work, and eventually conduce to that complete mastery over the subject, which every good teacher will aim at imparting, and every real student endeavour to attain. Since there is no 'royal road' to such a goal, and practice and familiarity with the work are the only training that can make perfect, the diligent study and imitation of a few good examples will neither be out of place nor useless.

(1.) Find the G.C.M. of $m^6 - m^5 + 2m^4 - m^3 + 2m^2 - m + 1$ and $m^6 - 1$.

$$m^6 - m^5 + 2m^4 - m^3 + 2m^2 - m + 1$$
$$= m^6 - m^5 + m^4 + m^4 - m^3 + m^2 + m^2 - m + 1$$
$$= m^4(m^2 - m + 1) + m^2(m^2 - m + 1) + (m^2 - m + 1)$$
$$= (m^2 - m + 1)(m^4 + m^2 + 1)$$
$$= (m^2 - m + 1)(m^2 + m + 1)(m^2 - m + 1).$$

Also, $m^6 - 1 = (m^3 - 1)(m^3 + 1)$

$$= (m - 1)(m^2 + m + 1)(m + 1)(m^2 - m + 1).$$

$\therefore$ G.C.M. $= (m^2 - m + 1)(m^2 + m + 1) = m^4 + m^2 + 1.$

(2.) Find the G.C.M. of $m^7 - 3m^6 + m^5 - 4m^2 + 12m - 4$ and $2m^4 - 6m^3 + 3m^2 - 3m + 1$.

$$m^7 - 3m^6 + m^5 - 4m^2 + 12m - 4$$
$$= m^5(m^2 - 3m + 1) - 4(m^2 - 3m + 1) = (m^5 - 4)(m^2 - 3m + 1).$$

Also, $2m^4 - 6m^3 + 3m^2 - 3m + 1$

$$= 2m^4 - 6m^3 + 2m^2 + m^2 - 3m + 1$$
$$= 2m^2(m^2 - 3m + 1) + (m^2 - 3m + 1)$$
$$= (2m^2 + 1)(m^2 - 3m + 1).$$

$\therefore$ G.C.M. $= m^2 - 3m + 1.$

N.B.—After the resolution of the first expression, it will be evident to the careful student that $m^2 - 3m + 1$, or unity, *must be the G.C.M.*; therefore the breaking up of the second quantity becomes a matter of increased simplicity.

(3.) Simplify $\dfrac{p^3 - 3pq^2 + 2q^3}{2p^3 + p^2q + pq^2 - 4q^3}$.

$$\frac{p^3 - 3pq^2 + 2q^3}{2p^3 + p^2q + pq^2 - 4q^3} = \frac{(p^3 - q^3) - 3pq^2 + 3q^3}{2p^3 - 2q^3 + p^2q + pq^2 - 2q^3}$$

$$= \frac{(p - q)(p^2 + pq + q^2) - 3q^2(p - q)}{2(p^3 - q^3) + q(p^2 + pq - 2q^2)}$$

$$= \frac{(p-q)(p^2+pq+q^2-3q^2)}{2(p-q)(p^2+pq+q^2)+q(p+2q)(p-q)}$$

$$= \frac{(p-q)(p^2+pq-2q^2)}{(p-q)(2p^2+2pq+2q^2+pq+2q^2)}$$

$$= \frac{p^2+pq-2q^2}{2p^2+3pq+4q^2} = \frac{(p+2q)(p-q)}{2p^2+3pq+4q^2}.$$

(4.) Find the L.C.M. of m^2-4n^2, $m^3+6m^2n+12mn^2+8n^3$, and $m^3-6m^2n+12mn^2-8n^3$

$m^2-4n^2=(m+2n)(m-2n)$;
$m^3+6m^2n+12mn^2+8n^3$
$=(m^3+8n^3)+6mn(m+2n)$
$=(m+2n)(m^2-2mn+4n^2)+6mn(m+2n)$
$=(m+2n)(m^2+4mn+4n^2)=(m+2n)(m+2n)^2$
$=(m+2n)^3$;
$m^3-6m^2n+12mn^2-8n^3$
$=(m^3-8n^3)-6mn(m-2n)$
$=(m-2n)(m^2+2mn+4n^2)-6mn(m-2n)$
$=(m-2n)(m^2-4mn+4n^2)$
$=(m-2n)(m-2n)^2=(m-2n)^3$.

Now it is evident, *from inspection*, that the L.C.M. of $(m-2n)(m+2n)$, $(m+2n)^3$, and $(m-2n)^3$,
$=(m+2n)^3(m-2n)^3=(m^2-4n^2)^3$.

(5.) Simplify $\dfrac{18(qr^2+p^2r+pq^2)-12(q^2r+pr^2+p^2q)-19pqr}{2p-3q}$.

$$\frac{18(qr^2+p^2r+pq^2)-12(q^2r+pr^2+p^2q)-19pqr}{2p-3q}$$

$$=\frac{18qr^2+18p^2r+18pq^2-12q^2r-12pr^2-12p^2q-19pqr}{2p-3q}$$

$$=\frac{-6r^2(2p-3q)-6pq(2p-3q)+r(18p^2-19pq-12q^2)}{2p-3q}$$

$$=\frac{-6r^2(2p-3q)-6pq(2p-3q)+r(2p-3q)\,(9p+4q)}{2p-3q}$$

$$=\frac{(2p-3q)(-6r^2-6pq+9pr+4qr)}{2p-3q}$$

$$=-6r^2-6pq+9pr+4qr.$$

(6.) Reduce $\dfrac{a^3+11a^2+30a}{9a^3+53a^2-9a-18}$ to lowest terms.

$$\frac{a^3+11a^2+30a}{9a^3+53a^2-9a-18}=\frac{a(a^2+11a+30)}{9a^3+53a^2-9a-18}$$

$$=\frac{a(a+5)(a+6)}{9a^3+54a^2-(a^2+9a+18)}=\frac{a(a+6)(a+5)}{9a^2(a+6)-(a+6)(a+3)}$$

$$=\frac{a(a+6)(a+5)}{(a+6)(9a^2-a-3)}=\frac{a(a+5)}{9a^2-a-3}.$$

N.B.—In the above example, the numerator resolves easily, the denominator is more difficult. As soon, however, as the factors of the numerator are obtained, it is at once evident that $a+6$ is the only possible *common* factor; and, therefore, any scheme which will get the $a+6$ contained in one part of the denominator, *must leave it as a factor in the remainder;* hence the resolution becomes both interesting and easy.

(7.) Similarly,

$$\frac{27a^5x^2-18a^4x^2-9a^3x^2}{36a^6x^2-18a^5x^2-27a^4x^2+9a^3x^2}$$

$$=\frac{9a^3x^2(3a^2-2a-1)}{9a^3x^2(4a^3-2a^2-3a+1)}$$

$$=\frac{(3a+1)(a-1)}{4a^3-4a^2+(2a^2-3a+1)}=\frac{(3a+1)(a-1)}{4a^2(a-1)+(a-1)(2a-1)}$$

$$=\frac{(3a+1)(a-1)}{(a-1)(4a^2+2a-1)}=\frac{3a+1}{4a^2+2a-1}.$$

(8.) Again,

$$\frac{p^4+p^2q^2+q^4}{p^4+2p^3q+3p^2q^2+2pq^3+q^4}$$

$$=\frac{(p^2+pq+q^2)(p^2-pq+q^2)}{p^4+p^2q^2+q^4+2p^3q+2p^2q^2+2pq^3}$$

$$=\frac{(p^2+pq+q^2)(p^2-pq+q^2)}{(p^2+pq+q^2)(p^2-pq+q^2)+2pq(p^2+pq+q^2)}$$

$$=\frac{(p^2+pq+q^2)(p^2-pq+q^2)}{(p^2+pq+q^2)(p^2+pq+q^2)}=\frac{p^2-pq+q^2}{p^2+pq+q^2}.$$

(9.) Find the G.C.M. of

$x^4-(p^2+1)x^2+p^2$ and $x^4-(p+1)^2x^2+2(p+1)px-p^2$.

$$x^4-(p^2+1)x^2+p^2=(x^2-p^2)(x^2-1)$$
$$=(x-p)(x+p)(x-1)(x+1);$$
$$x^4-(p+1)^2x^2+2(p+1)px-p^2$$
$$=x^4-\{(p+1)^2x^2-2(p+1)px+p^2\}$$
$$=x^4-\{(p+1)x-p\}^2$$
$$=\{x^2-(p+1)x+p\}\{x^2+(p+1)x-p\}$$
$$=(x-p)(x-1)\{x^2+(p+1)x-p\};$$
$$\therefore \text{ G.C.M.}=(x-p)(x-1).$$

(10.) Also,

$$\frac{2p^3+13p^2q-15pq^2-126q^3}{p^2q+6pq^2-7q^3}\times\frac{2p^3+19p^2q+35pq^2}{p^3-p^2q-4pq^2-6q^3}$$

$$\div\frac{2p^2+5pq}{pq-q^2}$$

$$=\frac{(2p^3+7p^2q)+(6p^2q-15pq^2-126q^3)}{q(p^2+6pq-7q^2)}\times$$

$$\frac{p(2p^2+19pq+35q^2)}{(p^3-3p^2q)+(2p^2q-4pq^2-6q^3)}\times\frac{q(p-q)}{p(2p+5q)}$$

$$=\frac{p^2(2p+7q)+q(6p^2-15pq-126q^2)}{q(p+7q)(p-q)}$$

$$\times\frac{p(p+7q)(2p+5q)}{p^2(p-3q)+2q(p^2-2pq-3q^2)}\times\frac{q(p-q)}{p(2p+5q)}$$

$$=\frac{p^2(2p+7q)+q(2p+7q)(3p-18q)}{p^2(p-3q)+2q(p-3q)(p+q)}$$

$$=\frac{(2p+7q)(p^2+3pq-18q^2)}{(p-3q)(p^2+2pq+2q^2}$$

$$=\frac{(2p+7q)(p+6q)(p-3q)}{(p-3q)(p^2+2pq+2q^2)}=\frac{(2p+7q)(p+6q)}{p^2+2pq+2q^2}.$$

(11.) $$\left(\frac{c-b}{c+b}-\frac{c^3-b^3}{c^3+b^3}\right)\div\left(\frac{c+b}{c-b}+\frac{c^2+b^2}{c^2-b^2}\right)$$

$$=\frac{\dfrac{(c-b)(c^2-cb+b^2)-(c-b)(c^2+cb+b^2)}{(c+b)(c^2-cb+b^2)}}{\dfrac{(c+b)^2+c^2+b^2}{(c-b)(c+b)}}$$

$$=\frac{\dfrac{(c-b)(c^2-cb+b^2-c^2-cb-b^2)}{c^2-cb+b^2}}{\dfrac{2(c^2+cb+b^2)}{c-b}}$$

$$=\frac{-2cb(c-b)^2}{2(c^2+cb+b^2)(c^2-cb+b^2)}=\frac{-cb(c-b)^2}{c^4+c^2b^2+b^4}$$

(12.) $$\frac{m^5-m^4n-mn^4+n^5}{m^4-m^3n-m^2n^2+mn^3}=\frac{m^4(m-n)-n^4(m-n)}{m^3(m-n)-mn^2(m-n)}$$

$$=\frac{(m-n)(m^4-n^4)}{(m-n)(m^3-mn^2)}=\frac{(m^2+n^2)(m^2-n^2)}{m(m^2-n^2)}=\frac{m^2+n^2}{m}.$$

(13.)
$$\frac{a}{1-\dfrac{a}{1+a+\dfrac{a}{1-a+a^2}}}=\frac{a}{1-\dfrac{a}{\dfrac{(a+1)(a^2-a+1)+a}{a^2-a+1}}}$$

$$=\frac{a}{1-\dfrac{\dfrac{a}{1}}{\dfrac{a^3+a+1}{a^2-a+1}}}=\frac{a}{1-\dfrac{a^3-a^2+a}{a^3+a+1}}$$

$$=\frac{a}{\dfrac{a^3+a+1-a^3+a^2-a}{a^3+a+1}}=\frac{\dfrac{a}{1}}{\dfrac{a^2+1}{a^3+a+1}}=\frac{a(a^3+a+1)}{a^2+1}.$$

71. A few illustrative Simple Equations will conclude this set of examples; and the student is desired to see clearly that each step is easily deducible from the one immediately preceding it.

(1.)
$$(x-a)(x-b)=(x-a-b)^2$$
$$(x-a)(x-b)=(x-a)^2-2b(x-a)+b^2$$
$$(x-a)(x-b-x+a+2b)=b^2$$
$$(x-a)(a+b)=b^2$$
$$ax+bx-a^2-ab=b^2$$
$$x(a+b)=a^2+ab+b^2$$
$$\therefore\ x=\frac{a^2+ab+b^2}{a+b}.$$

(2.)
$$\frac{1}{x-a}-\frac{1}{x-b}=\frac{a-b}{x^2-ab}$$
$$(x^2-ab)(x-b)-(x-a)(x^2-ab)=(a-b)(x-a)(x-b)$$

$$(x^2 - ab)(x - b - x + a) = (a - b)(x - a)(x - b)$$
$$(x^2 - ab)(a - b) = (a - b)(x - a)(x - b)$$
$$x^2 - ab = x^2 - bx - ax + ab$$
$$x(a + b) = 2ab$$
$$\therefore x = \frac{2ab}{a+b}.$$

(3.) $(y+1)(y+2)(y+3) = (y-1)(y-2)(y-3)$
$+3(4y-2)(y+1)$

$$y^3 + 6y^2 + 11y + 6 = y^3 - 6y^2 + 11y - 6 + 12y^2 + 6y - 6$$
$$6y = 18;\quad \therefore y = 3^{\circ}.$$

(4.) $(x-9)(x-5)(x-1) = (x-2)(x-4)(x-6)$
$(x-10)$

$$x^4 - 22x^3 + 164x^2 - 458x + 315$$
$$= x^4 - 22x^3 + 164x^2 - 488x + 480$$
$$30x = 165;\ 2x = 11;\ \therefore x = 5\tfrac{1}{2}^{\circ}.$$

(5.) $\dfrac{3-2x}{1-2x} - \dfrac{2x-5}{2x-7} = 1 - \dfrac{4x^2-1}{7-16x+4x^2}$

$$\frac{2x-3}{2x-1} - \frac{2x-5}{2x-7} = \frac{7 - 16x + 4x^2 - 4x^2 + 1}{4x^2 - 16x + 7}$$
$$= \frac{-8(2x-1)}{(2x-7)(2x-1)}$$
$$\frac{2x-3}{2x-1} - \frac{2x-5}{2x-7} = \frac{-8}{2x-7}$$
$$\frac{2x-3}{2x-1} = \frac{2x-13}{2x-7}$$
$$(2x-3)(2x-7) = (2x-13)(2x-1)$$
$$4x^2 - 20x + 21 = 4x^2 - 28x + 13$$
$$8x = -8, \therefore x = -1.$$

Or, thus,

$$\frac{3-2x}{1-2x} - \frac{2x-5}{2x-7} = 1 - \frac{4x^2-1}{7-16x+4x^2}$$

This Solution is founded on the theory contained in Article 60.

$$\frac{2x-3}{2x-1}-\frac{2x-5}{2x-7}=1-\frac{(2x+1)(2x-1)}{(2x-7)(2x-1)}$$

$$(2x-3)(2x-7)-(2x-5)(2x-1)=(2x-7)$$
$$(2x-1)-(2x+1)(2x-1)$$

$$(2x-7)(2x-3-2x+1)=(2x-1)(2x-5-2x-1)$$
$$2(2x-7)=6(2x-1)$$
$$2x-7=6x-3;\ 4x=-4;\ \therefore x=-1.$$

(6.) $$\frac{y-n}{y-n-1}-\frac{y-n-1}{y-n-2}=\frac{y-m}{y-m-1}-\frac{y-m-1}{y-m-2}$$

$$\frac{(y-n)(y-n-2)-(y-n-1)^2}{(y-n-1)(y-n-2)}=$$
$$\frac{(y-m)(y-m-2)-(y-m-1)^2}{(y-m-1)(y-m-2)}$$

$$\frac{(y-n)^2-2(y-n)-(y-n)^2+2(y-n)-1}{(y-n-1)(y-n-2)}$$
$$=\frac{(y-m)^2-2(y-m)-(y-m)^2+2(y-m)-1}{(y-m-1)(y-m-2)}$$

$$\frac{-1}{(y-n)^2-3(y-n)+2}=\frac{-1}{(y-m)^2-3(y-m)+2}$$

$$y^2-2ny+n^2-3y+3n+2=y^2-2my+m^2-3y+$$
$$3m+2$$

$$2y(m-n)=(m-n)(m+n)+3(m-n)$$

$$\therefore y=\frac{m+n+3}{2}.$$

(7.) $\cdot5x-2=\cdot25x+\cdot2x-1$

$$\frac{5x}{10}-2=\frac{25x}{100}+\frac{2x}{10}-1$$

$$\frac{x}{2}-2=\frac{x}{4}+\frac{x}{5}-1$$

$$10x - 40 = 5x + 4x - 20$$
$$\therefore x = 20.$$

(8.) $(x+a)(2x+b+c)^2 = (x+b)(2x+a+c)^2$

$$(x+a)\{(2x+c)^2 + 2b(2x+c) + b^2\}$$
$$= (x+b)\{(2x+c)^2 + 2a(2x+c) + a^2\}$$
$$(2x+c)^2(x+a-x-b) + 2(2x+c)(bx+ab-ax-ab)$$
$$= a^2x + a^2b - b^2x - ab^2$$
$$(2x+c)^2(a-b) - 2x(2x+c)(a-b) = x(a+b)(a-b)$$
$$+ab(a-b)$$
$$4x^2 + 4cx + c^2 - 4x^2 - 2cx = ax + bx + ab$$
$$x(2c-a-b) = ab - c^2$$
$$\therefore x = \frac{ab-c^2}{2c-a-b}.$$

(9.) $y - 3 - (3-y)(y+1) = (y-3)(y+1) - (y-3)$

$$2(y-3) = 0$$
$$\therefore y = 3.$$

(10.) $\surd(6x) + \surd(6x+45) = 15$

$$\surd(6x+45) = 15 - \surd(6x)$$
$$6x + 45 = 225 - 30\surd(6x) + 6x$$
$$30\surd(6x) = 180$$
$$\surd(6x) = 6$$
$$6x = 36$$
$$\therefore x = 6.$$

(11.) $\surd(a+14) + \surd(a-14) = 14$

$$\surd(a+14) = 14 - \surd(a-14)$$
$$a + 14 = 196 - 28\surd(a-14) + a - 14$$
$$28\surd(a-14) = 168$$
$$\surd(a-14) = 6$$
$$a - 14 = 36$$
$$\therefore a = 50.$$

(12.) $\surd(y-n) + \surd(y-m) = \surd(n-m)$

$$y - n + 2\surd(y-n)\surd(y-m) + y - m = n$$

$2\sqrt{(y-n)}\sqrt{(y-m)} = 2n - 2y$
$y^2 - my - ny + mn = n^2 - 2ny + y^2$
$ny - my = n^2 - mn$
$y(n-m) = n(n-m)$
$\therefore y = n.$

72. As the student advances from the more elementary parts of his Algebra, he will find the foregoing formulæ and remarks continually coming to his aid; and, having thoroughly understood them, he will be able the more readily to comprehend and apply the many theorems that will, after this, be presented to his notice. Let him carefully and completely master the contents of this little work, and he will not only have made great progress in the subject, but will also have the satisfaction of knowing that he has begun it in the only way in which Algebra can possibly be made a *scientific* and delightful study.

ANSWERS TO EXAMPLES.

CHAPTER II.

1. a^2+c^2+2ac.
2. d^2+e^2+2de.
3. g^2+h^2+2gh.
4. m^2+n^2+2mn.
5. p^2+q^2+2pq.
6. m^2+q^2+2mq.
7. $169+81+234=484$.
8. $121+64+176=361$.
9. $289+324+612=1225$.
10. $225x^2+169y^2+390xy$.
11. $49y^4+64z^2+112y^2z$.
12. $a^2x^2+b^2y^2+2abxy$.
13. $p^2q^2+4s^2+4pqs$.
14. $x^4+y^4+2x^2y^2$.
15. $x^6+4y^6+4x^3y^3$.
16. $36x^2y^2+16a^2b^2+48abxy$.
17. $a^2b^2+c^2d^2-2abcd$.
18. $x^6+y^6-2x^3y^3$.

19. $x^2y^2+z^2-2xyz.$

20. $y^2+x^2z^2-2xyz.$

21. $4z^2+16y^2+16yz.$

22. $16b^2+36c^2d^2+48bcd.$

23. $m^2p^2+4n^2q^2-4mnpq.$

24. $49m^2x^2+9s^2t^2+42mstx.$

25. $a^2+b^2+c^4+2ab-2ac^2-2bc^2.$

26. $a^2+x^2+4b^2+4y^2+2ax-4ab-4ay-4bx-4xy+8by.$

27. $x^2+y^2+36+z^2-2xy-12x+2xz+12y-2yz-12z.$

28. $4a^2+9b^2+16c^2+25d^2+12ab+16ac+20ad+24bc$
$+30bd+40cd.$

29. $p^2+4q^2+r^2+9m^2-4pq+2pr-6mp-4qr+12mq$
$-6mr.$

30. $1-4a+10a^2-16a^3+19a^4-16a^5+10a^6-4a^7+a^8.$

31. $x^8+2x^7y-x^6y^2-4x^5y^3+3x^4y^4+6x^3y^5-3x^2y^6$
$-4xy^7+4y^8.$

32. $a^4+4a^3x+6a^2x^2+4ax^3+x^4.$

33. $16b^4+96b^3c+216b^2c^2+216bc^3+81c^4.$

34. $x^8+8x^7y+28x^6y^2+56x^5y^3+70x^4y^4+56x^3y^5$
$+28x^2y^6+8xy^7+y^8.$

35. $4a^2+4b^2+c^2+d^2+8ab-4ac+4ad-4bc+4bd-2cd.$

36. $b^2+c^2+d^2+x^2+y^2+2bc+2bd+2bx-2by+2cd+2cx$
$-2cy+2dx-2dy-2xy.$

37. $\frac{x^2}{a^2}+\frac{a^2}{x^2}+2.$

38. $3+\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{2x}{y}+\frac{2y}{x}.$

39. $\left(\frac{x+a}{x-a}\right)^2+\left(\frac{x-a}{a+a}\right)^2+2.$

CHAPTER III.

1. $s^2 - 4t^2$.
2. $x^4 - c^4$.
3. $x^4 - 16r^4$.
4. $4p^2 - q^2$.
5. $x(a^4 - x^4)$.
6. $\frac{49x^2}{4b^2} - \frac{9y^2}{16c^2}$.
7. $a^8 + x^4 - 2a^4x^2$.
8. $x^{16} + a^{16} - 2a^8x^8$.
9. $(x^2+9y)(x^2-9y)$.
10. $(x^4+1)(x^2+1)(x+1)(x-1)$.
11. $(x^2-8)(x^2+8)$.
12. $(1-15x^2)(1+15x^2)$.
13. $\left(\frac{x^2}{a}+\frac{y^2}{b}\right)\left(\frac{x^2}{a}-\frac{y^2}{b}\right)$.
14. $(a-b)(a-b)(a+b)(a+b)$.
15. $a^2(x+y)(x-y)$.
16. $(\frac{1}{2}a - \frac{1}{3}b)(\frac{1}{2}a+\frac{1}{3}b)$.
17. $(x^2y^2z^2+1)(xyz+1)(xyz-1)$.
18. $(16x^2+25y^2)(4x+5y)(4x-5y)$.
19. $(a^2b^6+c^4)(ab^3+c^2)(ab^3-c^2)$.
20. $ap(m+n^2)(m-n^2)$.
21. $(y-2)(y+2)(y^2+4)(y+3)(y-3)=(y^2-4)$
$(y^2+4)(y^2-9)=(y^4-16)(y^2-9)$.
22. $c^4 - a^4 + 4a^3b - 6a^2b^2 + 4ab^3 - b^4$.

23. $\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\left({}^{a}-a\right).$

24. $a^8+b^8-2a^4b^4.$

25. $\frac{1}{a^8+b^8+a^4b^4}.$

26. $x^4-y^4.$

27. $(x^8+256)(x^4+16)(x^2+4)(x+2)(x-2).$

28. $1-\frac{x^2}{1-2x+x^2}=\frac{1-2x}{1-2x+x^2}=\frac{1-2x}{(1-x)^2}.$

29. $\frac{x^2+y^2}{x}.$

30. $(a+x)^2-\frac{1}{(a+x)^2}.$

31. $\frac{a^4}{p_4}-\frac{x^4}{y^4}-1+\frac{2x^2}{y^2}.$

32. $-9.$

33. $a^2+b^2+ab.$

34. $\{(a+b)^{2n}+c^{2n}\}\{(a+b)^n+c^n\}\{(a+b)^n-c^n\}.$

CHAPTERS IV. & V.

1. $(m^4+n^4)(m^2+n^2)(m-n)$.

2. $a^{18}-b^9=(a^6)^3-(b^3)^3=(a^6-b^3)(a^{12}+a^6b^3+b^6)$
$=\{(a^2)^3-b^3\}(a^{12}+a^6b^3+b^6)$
$=(a^2-b)(a^4+a^2b+b^2)(a^{12}+a^6b^3+b^6)$.

3. $$\frac{a^{18}+b^9}{a^4-a^2b+b^2}=\frac{(a^6+b^3)(a^{12}-a^6b^3+b^6)}{a^4-a^2b+b^2}$$
$$=\frac{(a^2+b)(a^4-a^2b+b^2)(a^{12}-a^6b^3+b^6)}{a^4-a^2b+b^2}$$
$$=(a^2+b)(a^{12}-a^6b^3+b^6).$$

4. $$\frac{(p^{12}-2p^6q^6+q^{12})(p^{12}+2p^6q^6+q^{12})}{(p^6-2p^3q^3+q^6)(p^6+2p^3q^3+q^6)(p^4+2p^2q^2+q^4)}$$
$$=\frac{(p^6-q^6)^2(p^6+q^6)^2}{(p^3-q^3)^2(p^3+q^3)^2(p^2+q^2)^2}$$
$$=\frac{(p^6-q^6)^2(p^2+q^2)^2(p^4-p^2q^2+q^4)^2}{(p^6-q^6)^2(p^2+q^2)^2}$$
$$=(p^4-p^2q^2+q^4)^2.$$

5. $$\frac{x^6+3x^4y^2+3x^2y^4+y^6+m^6}{m^2+y^2+x^2}=\frac{(x^2+y^2)^3+(m^2)^3}{x^2+y^2+m^2}$$
$$=\frac{(x^2+y^2+m^2)(x^4+2x^2y^2+y^4-m^2x^2-m^2y^2+m^4)}{x^2+y^2+m^2}$$
$$=m^4+x^2(x^2+y^2-m^2)+y^2(x^2+y^2-m^2)$$
$$=m^4+(x^2+y^2-m^2)(x^2+y^2).$$

6. $a(a+b)$.

7 x^2-y^2.

8. $$\frac{m^6+3m^4n^2+3m^2n^4+n^6-a^6+3a^4x^2-3a^2x^4+x}{m^2+n^2-a^2+x^2}$$

$$=\frac{(m^2+n^2)^3-(a^2-x^2)^3}{m^2+n^2-a^2+x^2}$$

$$[+(a^2-x^2)^2\}$$

$$=\frac{(m^2+n^2-a^2+x^2)\{(m^2+n^2)^2-(m^2+n^2)(a^2-x^2)}{m^2+n^2-a^2+x^2}$$

$$=(m^2+n^2)^2-(m^2+n^2)(a^2-x^2)+(a^2-x^2)^2.$$

9. $$\frac{8m^3+x^3+3x^2y+3xy^2+y^3}{y+2m+x}=\frac{(2m)^3+(x+y)^3}{2m+x+y}$$

$$=\frac{(2m+x+y)\{(2m)^2-2m(x+y)+(x+y)^2\}}{2m+x+y}$$

$$=4m^2-2m(x+y)+(x+y)^2.$$

10. $(pq)^{12}-(pq)^{11}mn+(pq)^{10}(mn)^2-(pq)^9(mn)^3$
$+(pq)^8(mn)^4$
$-(pq)^7(mn)^5+(pq)^6(mn)^6-(pq)^5(mn)^7+(pq)^4$
$(mn)^8$
$-(pq)^3(mn)^9+(pq)^2(mn)^{10}-(pq)(mn)^{11}+(mn)^{12}.$

11. $729x^{21}+343q^{27}=(9x^7)^3+(7q^9)^3$
$=(9x^7+7q^9)(81x^{14}-63x^7q^9+49q^{18}).$

12. $(x+y)^3-(x+y-z)^3$
$=(x+y-x-y+z)\{(x+y)^2+(x+y)(x+y-z)+$
$(x+y-z)^2\}$
$=z\{(x+y)^2+(x+y)^2-(x+y)z+(x+y)^2-2(x+y)$
$z+z^2\}$
$=z\{3(x+y)^2-3(x+y)z+z^2\};$

or

$=(x+y)^3-\{(x+y)-z\}^3$
$=(x+y)^3-(x+y)^3+3(x+y)^2z-3(x+y)z^2+z^3$
$=z\{3(x+y)^2-3(x+y)z+z^2\}.$

13. $2(x^3+x^2y+xy^2)-x^3+y^3$
$=2x(x^2+xy+y^2)-(x^3-y^3)$
$=2x(x^2+xy+y^2)-(x-y)(x^2+xy+y^2)$
$=(x^2+xy+y^2)(2x-x+y)=(x+y)(x^2+xy+y^2).$

14. $\{(p-q)-(p+q)\}^3 = (-2q)^3 = -8q^3$.

15. $kl(k-l)$.

16. $m^{18}-n^{12} = (m^9+n^6)(m^9-n^6)$
$= (m^3+n^2)(m^6-m^3n^2+n^4)(m^3-n^2)(m^6+m^3n^2$
$+n^4)$.

17. $(a+b+c)^6 = \{(a+b+c)^3\}^2 = [\{(a+b)+c\}^3]^2$
$= (a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2$
$+3bc^2+c^3)^2$.

18. $(a+2b)^3+3(a+2b)^2(3c+4d)+3(a+2b)(3c+4d)^2$
$+(3c+4d)^3$
$=a^3+6a^2b+12ab^2+8b^3+3(a^2+4ab+4b^2)(3c+4d)$
$+3(a+2b)(9c^2+24cd+16d^2)+27c^3+108c^2d+144$
cd^2+64d^3.

19. $17^3+13^3 = (17+13)\{(17)^2-17\times13+(13)^2\}$
$= 30(289-221+169) = 30\times237 = 7110$.

20. $19^3-14^3 = (19-14)\{(19)^2+19\times14+(14)^2\}$
$= 5(361+266+196) = 5\times823 = 4115$.

CHAPTER VI.

1. $(m+15n)(m+14n)$.
2. $(xy+19z)(xy+18z)$.
3. $(p^2q^2-13r)(p^2q^2-8r)$.
4. $(a^4z-p^2)(a^4z-18p^2)$.
5. $(a^4z+p^2)(a^4z-20p^2)$.
6. $(b^3y^2z+22q^2)(b^3y^2z-11q^2)$.
7. $(a+m)(a+n)$.
8. $(p+q)^2+13(p+q)(x+y)+42(x+y)^2$
   $=\{(p+q)+6(x+y)\}\{(p+q)+7(x+y)\}$
   $=(p+q+6x+6y)(p+q+7x+7y)$.
9. $(3am-12q)(am-10q)$.
10. $\{2(m+n)-(x+y)\}\{(m+n)-3(x+y)\}$
    $=(2m+2n-x-y)(m+n-3x-3y)$.
11. $(pq-xy)(pq-129xy)$.
12. $(x-3)+(x-3)(x+1)=x(x-3)+8$,
    $(x-3)(1+x+1-x)=8$
    $x-3=4;\quad \therefore\ x=7$.
13. $x(\frac{1}{3}-\frac{1}{4}-\frac{1}{5}+\frac{1}{6})=(\frac{1}{3}-\frac{1}{4}-\frac{1}{5}+\frac{1}{6})$;
    $\therefore\ x=1$.
14. $\dfrac{(x+9)(x+4)}{(x+9)(x+7)}=\dfrac{x+4}{x+7}$.
15. $\dfrac{1}{(p-a)(p-b)}+\dfrac{1}{(p-a)(p-c)}+\dfrac{1}{(p-b)(p-c)}$;

$$=\frac{p-c+p-b+p-a}{(p-a)(p-b)(p-c)}$$

$$=\frac{3p-a-b-c}{(p-a)(p-b)(p-c)}.$$

16. $$\frac{(3m^2+n^2)(4m^2-n^2)}{(3m^2+n^2)(3m^2+n^2)}=\frac{4m^2-n^2}{3m^2+n^2}.$$

17. $$\frac{1}{(x-y)(x-z)}-\frac{1}{(x-y)(y-z)}+\frac{1}{(x-z)(y-z)}$$
$$\frac{y-z-x+z+x-z}{(x-y)(x-z)(y-z)}=0.$$

18. $$\frac{(p-m)(p-n)}{(p-m)(p-q)}=\frac{p-n}{p-q}.$$

19. $$\frac{(2ax+y)(ax+2y)}{(2ax+y)(3ax+2y)}=\frac{ax+2y}{3ax+2y}.$$

20. $$\frac{p(p^3-6p^2y+12py^2-8y^3)-y(y^3-6py^2+12p^2y-8p^3)}{(p+y)^3}$$
$$=\frac{p^4-6p^3y+12p^2y^2-8py^3-y^4+6py^3-12p^2y^2+8p^3y}{(p+y)^3} \qquad [-y^2)$$
$$=\frac{p^4-y^4+2p^3y-2py^3}{(p+y)^3}=\frac{(p^2-y^2)(p^2+y^2)+2py(p^2}{(p+y^3)}$$
$$=\frac{(p^2-y^2)(p^2+2py+y^2)}{(p+y)^3}=\frac{(p-y)(p+y)^3}{(p+p)^3}$$
$$=p-y.$$

21 $$\frac{\{(x+y)-5c\}\{(x+y)-6c\}}{x+y-5c}=x+y-6c.$$

22. See Art. 54 (*b*), (2).

23. See Art. 54 (1), and Art. 54 (*a*).

24. See Art. 55 (5).

25. $$\{(x+y)+z\}^3-\{(x+y)-z\}^3-\{-(x-y)+z\}^3-\{(x-y)+z\}^1$$
$$=\left\{\begin{array}{l} \quad (x+y)^3+3(x+y)^2z+3(x+y)z^2+z^3 \\ -(x+y)^3+3(x+y)^2z-3(x+y)z^2+z^3 \\ +(x-y)^3-3(x-y)^2z+3(x-y)z^2-z^3 \\ -(x-y)^3-3(x-y)^2z-3(x-y)z^2-z^3 \end{array}\right.$$
$$=6(x^2+2xy+y^2-x^2+2xy-y^2)z=24xyz.$$

26. See Art. 55 (2).

27. See Art. 54 (*b*), (7).

28. $(x-6)(x-7)$; $(x-2)(x-7)$; $(x-5)(x-7)$;
$\therefore$ L.C.M. $=(x-2)(x-5)(x-6)(x-7)$.

29. $x^4-144=(x^2-12)(x^2+12)$;
$x^4-25x^2+156=(x^2-12)(x^2-13)$;
$x^4-5x^2-84=(x^2-12)(x^2+7)$;
$\therefore$ L.C.M. $=(x^2+7)(x^2+12)(x^2-12)(x^2-13)$.

30. $(p^3+19)(p^3+1)$; $(p^3+19)(p^3-19)$; $p^2(p^3+19)$;
$q(p^3+19)(p^3-1)$;
$\therefore$ L.C.M. $p^2q(p^3+1)(p^3-1)(p^3+19)(p^3-19)$

31. $(2k+3y)(2k-3y)$; $(2k+y)(2k-y)$;
$(2k-3y)(2k-y)$; $(2k-y)(2k+3y)$;
$(2k+y)(2k-3y)$; $(2k+3y)(2k+y)$;
$\therefore$ L.C.M. $=(2k+3y)(2k-3y)(2k+y)(2k-y)$;
or $=(4k^2-9y^2)(4k^2-y^2$.

32. $(x-2)(x-2)$; $(x-2)(x-5)$;
$\therefore$ G.C.M. $=x-2$.

33. $2(x^2+12)(x^2-12)$; $y^2(x^2-12)(x^2+1)$;
$\therefore$ G.C.M. $=x^2-12$.

34. $(p+11q)(p+10q)$; $(p+10q)(p-9q)$;
$2(p-11q)(p+10q)$; $\therefore$ G.C.M. $=p+10q$.

CHAPTER VII.

1. $a^4+4a^3y+6a^2y^2+4ay^3+y^4$.
2. $16a^4-32a^3y+24a^2y^2-8ay^3+y^4$. $[a^8$.
3. $82x^5+240ax^4+720a^2x^3+1080a^3x^2+810a^4x+243$
4. $a^5-20a^4x+160a^3x^2-640a^2x^3+1280ax^4-1024x^5$.
5. See $(a+b+c)^5$ on page 53.
6. $(x-y)^6+6(x-y))^5z+15(x-y)^4z^2+20(x-y)^3z^3$
$+15(x-y)^2z^4+6(x-y)z^5+z^6$. $[-b^7$.
7. $a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5+7ab^6$
8. $a^8-8a^7b+28a^6b^2-56a^5b^3+70a^4b^4-56a^3b^5+$
$28a^2b^6-8ab^7+b^8$.
9. $\{4a+(6b+1)\}^3=64a^3+48a^2(6b+1)+12a(6b+1)^2$
$+(6b+1)^3$
$=64a^3+288a^2b+48a^2+12a(36b^2+12b+1)+216b^3$
$+108b^2+18b+1$
$=64a^3+288a^2b+18a^2+432ab^2+144ab+12a+$
$216b^3+108b^2+18b+1$.
10. $y^7+21y^6+189y^5+945y^4+2835y^3+5103y^2+5103y$
$+2187$.
11. $m^9+9m^8n+36m^7n^2+84m^6n^3+126m^5n^4+126m^4n^5$
$+84m^3n^6+36m^2n^7+9mn^8+n^9$.
12. $m^{10}-10m^9n+45m^8n^2-120m^7n^3+210m^6n^4-$
$252m^5n^5$
$+210m^4n^6-120m^3n^7+45m^2n^8-10mn^9+n^{10}$.

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SPECIMEN PAGE.

THE EXPLANATORY POETICAL READER. 9

THE SOLDIER'S DREAM.—*Campbell.*

OUR bugles sang truce,[1] for the night cloud had lowered,[2]
And the sentinel stars set their watch in the sky;
And thousands had sunk on the ground overpowered—
The weary to sleep, and the wounded to die.

When reposing that night on my pallet of straw,
By the wolf-scaring fagot,[3] that guarded the slain,
At the dead of the night, a sweet vision[4] I saw,
And thrice, ere the morning, I dreamt it again.

Methought, from the battle-field's dreadful array,
Far, far I had roamed on a desolate[5] track;
'Twas autumn, and sunshine arose on the way
To the home of my fathers, that welcomed me back.

I flew to the pleasant fields traversed[6] so oft
In life's morning march,[7] when my bosom was young:
I heard my own mountain-goats bleating aloft,
And knew the sweet strain, that the corn-reapers sung.

Then pledged we the wine-cup, and fondly I swore
From my home and my weeping friends never to part:
My little ones kissed me a thousand times o'er,
And my wife sobbed aloud in her fulness of heart.

Stay, stay with us!—rest; thou art weary and worn:
And fain[8] was their war-broken soldier to stay—
But sorrow returned with the dawning of morn,
And the voice, in my dreaming ear, melted away.

—o—

[1] *Our bugles sang truce*, gave the signal to cease fighting for a time. A bugle is a musical instrument. [2] *Lowered* (pronounced so as to rhyme with overpowered, third line), appeared stormy, gloomy, or threatening. [3] *Wolf-scaring fagot*, fires lighted to frighten away the wolves. [4] *Vision*, a dream. [5] *Desolate*, dreary, solitary, uninhabited. [6] *Traversed*, wandered over. [7] *Life's morning march*, the days of childhood. [8] *Fain*, glad, happy.

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